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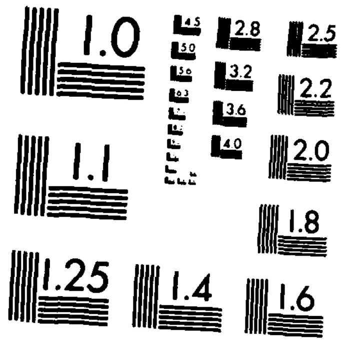

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BULK AVAILABILITY

BY

CLIFFORD MARSHALL

FEBRUARY 1983

OFFICE OF NAVAL RESEARCH

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Letter on file

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CHAPTER 1. BASIC CONCEPTS AND METHODOLOGY

Section 1.1 Introduction

The concept of availability represents the desire that a piece of equipment be operational when it is called upon to function. Availability extends the basic notion of reliability which in its pure meaning only deals with the continuing proper operation of a unit. Availability allows equipment to undergo repair or replacement either due to breakdown or as standard maintainance procedures. The concept of availability can be applied to a small unit, complex systems, or generalized to personnel. It therefore forms a major part in logistic analyses over a broad area of applications. A number of mathematical models have been developed for the analysis and design of availability requirements. An example of such mathematical approaches to some aspects of availability are given in (1) and (2) where a definition is developed for component availability which, unlike component reliability, must be placed within the context of the system in which a component is operating.

The field of contract structuring is also concerned with availability. Most contracts for the procurement of complex systems include specification of the availability of an operational system for a period following acceptance of the equipment itself. One approach to this aspect of contract structuring has been through the Reliability Improvement Warranty (RIW). By requiring the contractor to assume some form of responsiblity for equipment failure, the RIW gives a cost measure to reliability. The usual measurement of reliability in terms of probability concepts like mean time between failures is difficult to deal with in contract negotiations, acceptance tests, and disputes over peformance.

By introducing a cost measure for reliability, the concept becomes more clearly recognized as a real feature of contract responsibility and its dimensions can be spelled out in specific terms. Of course what is really desired is a warranty on availability, which includes both reliability concepts and aspects of the logistic support required for equipment supplied under contract. In (3) and (4) the ideas of availability warranties and incentive type contract structures have been given mathematical representation. That material illustrates the mathematical model approach to the analysis of availability warranties.

Most of the mathematical models and analyses of availability and availability (reliability) warranties have considered individual units of equipment or collections of such units in which the availability of each unit is desirable and important. One may refer to such considerations as "unit availability." The major mathematical tools for such studies are probability, statistics, and queuing theory. An illustration of this kind of analysis is given in (5). The advantage of probabilistic models is that a deterministic analysis of unit failure is in most cases too complex and the methodology of stochastic processes is well developed and yields useful results under many conditions. However, stochastic models have two important disadvantages: they require data for their useful implementation that are often very difficult or indeed impossible to obtain, and they are hard to interpret to people untrained in the details of probabilistic reasoning, for example as part of a contract negotiation. Thus it would be desirable to have alternative methodologies for availability models if they could be developed so as to give meaningful and useful analysis. This does not seem possible in most cases. However, for the concept of bulk availability, such an alternative methodology does

seem possible. This is the primary goal of the work described in the report: to define a non-probabilistic model for bulk availability. It is felt that development of such a model will show the feasibility of an alternative methodology for at least some types of availability analyses. It also provides a model that is useful directly for the study of bulk availability. In particular, it gives a basis for the formulation of availability warranties in contracts that deal with bulk availability.

Bulk availability differs from unit availabiltiy in that the individual units are not the primary consideration but rather the number of units that are available at a given time. A bulk availability model applies to situations in which there are a number of similar units making up the total system which is defined as the collection of units. Thus the system itself is unstructured in the model formulation. Availability of such a bulk system is defined in terms of the units that are available. Such models apply to a group of workers that are essentially interchangeable for functional purposes such as a platoon of army personnel. The platoon is operational if some specified percent of its composite personnel is operational. In the same way the model applies to fleets of trucks or aircraft, highly redundant groups of equipment, or gracefully degrading complex equipment.

Because of the definition of bulk availability as a fraction of the total collection of units being operational a deterministic model suggests itself as an alternative to the usual stochastic models of availability. In bulk availability the individual units are not important but only the number that are in a particular state of operation at a particular time. The model presented in this report employs transfer rates between states such that the actual number of units changing state depend on

these rates and the number of units in a state. Other model definitions are also possible and may be more appropriate in some situations. However, the present model, as defined in the next section, applies to situations in which the number of transfers from one state to another depend on the number of units in a state and the transfer rate. For example in a platoon that is exposed to a sickness, the number of personnel who become ill in each time period depends on the rate of contagion and on the number still unaffected who are therefore able to become ill.

Section 6 employs the bulk availability model to describe a methodology for the analysis of bulk availability warranties and related logistic considerations. This material is based on (6) while this report gives some additional background and extensions of the basic ideas given in (6).

Section 1.2 Definitions and Terminology

This study considers a model for bulk availability in which there are a large number of similar units that fail at a constant rate λ. Upon failure, a unit enters a waiting line and from there enters repair service. The service activity can accomodate up to r units and completes service at a constant rate μ. It is assumed that the number of units departing from a state is equal to the appropriate rate parameter times the number of units required or available depending on which state is under consideration. The model is constructed as a system of ordinary differential equations for the three quantities:

$m_a(t)$ = the number of units that are active (available),

$m_w(t)$ = the number of units that are waiting for service, and

$m_s(t)$ = the number of units in service.

All three quantities are functions of time as the independent variable.

In this model system, availability can be defined as a lower bound on $m_a(t)$.

It is assumed that the system is conservative in that $m_a(t) + m_w(t) + m_s(t)$ is a constant.

The differential equation system is intrinsically non-linear in that part of the forcing function depends on the values of the unknown functions for its formulation. Thus the model is formulated in two regions of the (m_s, m_w) plane. Figure 1 shows the relations between the three states of the system: available, waiting, and in service.

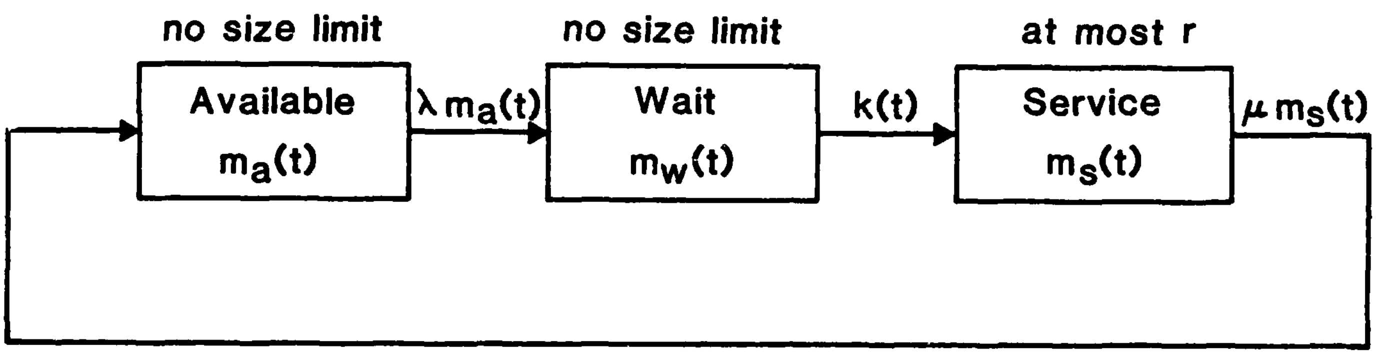

Figure 1
States of the Bulk Availability Model

The system of differential equations governing the flow of units through the states of the bulk availability model follows directly from the conservation and flow rate assumptions stated above. This system has the form:

$$\frac{dm_a(t)}{dt} = \mu m_s(t) - \lambda m_a(t)$$

$$\frac{dm_w(t)}{dt} = \lambda m_a(t) - K(t)$$

$$\frac{dm_s(t)}{dt} = K(t) - \mu m_s(t)$$

In this system, the quantity K(t) depends on the relative values of m_s and m_w leading to two specific forms for K(t) as given below. Because of the conservation assumption, the initial values $s_1 = m_s(t_o)$, $w_1 = m_w(t_o)$, and $a_1 = m_a(t_o)$ satisfy the condition $s_1 + w_1 + a_1 = m_s(t) + m_w(t) + m_a(t)$ for all values of t. Since the differential equation system must be followed across different forms the "initial" values a_1, w_1, and s_1 are incorporated as parameters into the solutions.

The value of K(t) depends on what is required by the service state and what is available from the waiting state. This leads to the following division of an (m_s, m_w) plane into two major regions.

<u>Region 1</u> $K = m_w(t)$, defined by $m_s \leqq r$ and $m_w < r - m_s + \mu m_s$.

<u>Region 2</u> $K = r - m_s(t) + \mu m_s(t)$, defined by $m_s \leqq r$ and

$$m_w \geqq r - m_s + \mu m_s.$$

These regions are shown in Figure 2.

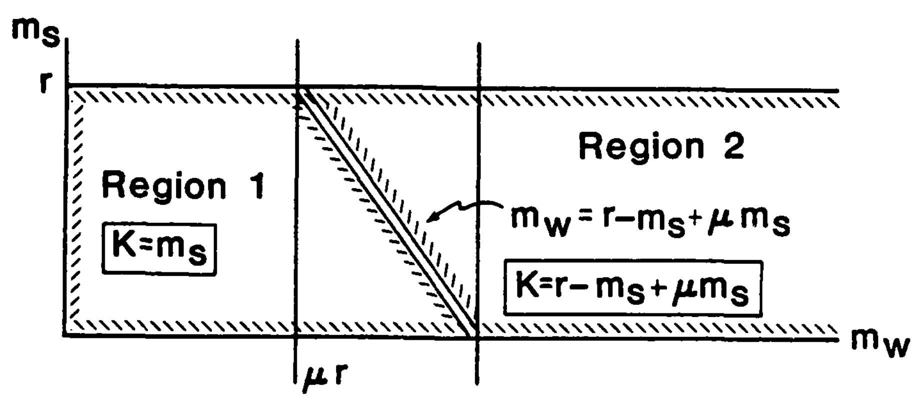

Figure 2
Regions Specifying Model Form

Solutions are obtained for Regions 1 and 2 in Sections 3 and 4 respectively. Some numerical examples are given in Section 5 to illustrate bulk availability analysis using the model described here.

One procedure for using the model would be to simulate the system using the differential equations to provide the model structure. This is equivalent to direct numerical solution of the equations. However, the system of differential equations can be solved directly in terms of explicit functions of time. This is the approach taken here and given in the following sections. The resulting functions of time are complicated and require computer evaluation as described in Section 5. Even so, direct explicit solutions are felt to be superior to numerical solutions of the differential equation system. In particular such explicit solutions can be studied directly for limiting values and special cases of the model parameters.

Section 1.3 Scope of Model Methodology

The deterministic mathematical model for bulk availability given in this paper relates the availability of a system to various system parameters. Unlike the usual stochastic model it does not depend upon an underlying probability framework and its results are specific quantities expressed as functions of time rather than expected values or other partial descriptions of random events. In this sense, the present bulk availability model is related to stochastic models in the way thermodynamics is related to statistical mechanics as a description of certain physical processes. The goal of thermodynamics is to relate various measurable quantities, e.g. temperature, to physical parameters without the necessity of a detailed analysis of the partical dynamics that produce such relations. This is also true of the bulk availability model in which the transition rates and initial distribution of units among states are system parameters. The model relates these parameters to the descriptive functions m_a, m_w, and m_s.

The value of the model is in its direct relation between a few system parameters and clearly identified descriptors of the system, particularly $m_a(t)$ which measures system availability. This direct relation is useful in contract negotiations, system design, logistic support design and implementation, and the conducting of acceptance test procedures. Stochastic models are much harder to interpret for use in all these aspects of contracting activity. However, this valuable feature of the bulk availability model is obtained at the expense of rather limited applicability. For the model to be meaningful, the assumptions required for the model must hold. This restricts model applicability to the bulk case in which the system description can indeed be given by a division of units into the three states of the model. Thus the model takes no account of any interaction between units or differences in utility between units. The conservation assumption also limits the model. There are only the three specified states so that any case in which a unit fails and cannot be repaired or replaced is not included in the present form of the model. Of course such additional states can be included in the model. The desirability of doing so to extend model validity must be balanced against the desire for relatively simple system descriptors which motivated this kind of model from the beginning.

The most serious limitation of the model is the assumption that the number of units that transfer state can be expressed as a product of a transfer rate and the amount available for transfer. In any particular case, this may or may not be a valid assumption. Its validity can be established by theoretical arguments or assumptions or by experimental data. If it cannot be validated in some way, it is likely that the par-

ticular bulk availability model presented here does not apply and should not be used. Other assumptions can be made in a deterministic model context, but the ones used here seem to give the most direct example of the kind of model of interest in this study.

CHAPTER 2. REGION 1 SOLUTIONS

Section 2.1 General Form of Region 1 Solutions

As specified in Section 1.2, the Region 1 form of the bulk availability model is given by the value $K(t) = m_w(t)$. This results in the following system of differential equations:

$$\dot{m}_a(t) = \mu m_s(t) - \lambda m_a(t),$$

$$\dot{m}_w(t) = \lambda m_a(t) - m_w(t),$$

$$\dot{m}_s(t) = m_w(t) - \mu m_s(t),$$

where the dot notation is used to indicated differentiation with respect to time. The initial conditions for the system are values $m_a(t_o)$, $m_w(t_o)$, and $m_s(t_o)$ specified at a time t_o which represents the time at which this system starts to govern the behavior of the model. Over time, the model may change from a Region 1 to a Region 2 system or conversely. Therefore, the initial conditions for a system, including the initial time may be obtained as specified data or as the final solutions of a previous solution form. This is described more fully in Section 4 on numerical solution procedures.

The method of Laplace transforms is used to obtain the Region 1 solutions. If $L[g(t)] \equiv G(s)$ is the Laplace transform of a function $g(t)$, define: $L[m_a(t)] = F_a(s)$, $L[m_w(t)] = F_w(s)$, and $L[m_s(t)] = F_s(s)$.

Also define the initial values as: $m_a(t_o) = a_1$, $m_w(t_o) = w_1$, and $m_s(t_o) = s_1$. The transformed equations for the Region 1 system have the following standard form:

$$
\begin{aligned}
(s+\lambda)F_a \qquad\qquad & - \mu F_s &= a_1 \\
-\lambda F_a + (s+1)F_w \qquad & &= w_1 \\
-F_w & + (s+\mu)F_s &= s_1
\end{aligned}
$$

This transformed system can be solved in the form:

$$F_a = \frac{\mu}{s+\lambda}F_s + \frac{a_1}{s+\lambda},$$

$$F_w = \frac{\lambda\mu}{(s+1)(s+\lambda)} F_s + \frac{a_1}{(s+1)(s+\lambda)} \lambda + \frac{w_1}{s+1},$$

where

$$F_s = \frac{s_1((s+1)(s+\lambda)}{P(s)} + \frac{w_1(s+\lambda)}{P(s)} + \frac{a_1\lambda}{P(s)},$$

and $P(s) = sQ(s)$, where $Q(s) = s^2 + (1+\lambda+\mu)s + \lambda + \mu + \mu\lambda$.

The form of the solution depends on the roots of the quadratic function $Q(s)$. The descriminant for this quadratic is:

$$D = (1-\lambda)^2 - 2\mu(1+\lambda) + \mu^2$$

or alternatively:

$$D = (1-\lambda-\mu)^2 - 4\mu\lambda.$$

The curve $D = 0$ is a rotated parabola symmetric about the line $\lambda = \mu$ in the (μ,λ) plane as shown in Figure 3.

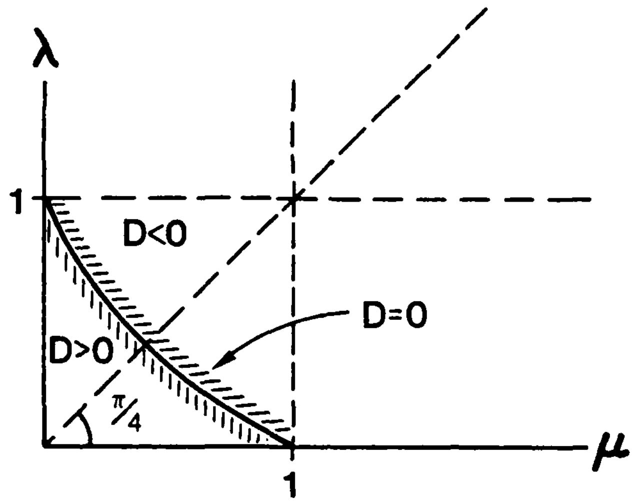

Figure 3
Areas of (μ,λ) Plane Defining D Values

A more detailed plot of the curve D = 0 is shown in Figure 4. The region with D < 0 is inside the parabola as shown. Any combination of μ and λ, within this region, gives a Case 1 solution. Values outside the parabola, where D > 0, correspond to Case 3 solutions. In the model as presently formulated μ and λ assume fixed values in an analysis. However, in generalizing the model or in considering the kind of results an analysis would yield it is interesting to consider linear variation of the form $\mu = K\lambda$. Straight lines such as these intersect the parabola in two points given by the λ values $\lambda_K = \frac{(1+K)\pm 2\sqrt{K}}{(1-K)^2}$. When K = 1, there is only one solution, K > 1 gives solutions on the lower branch and K < 1 gives solutions on the upper branch of the parabola respectively. A typical intersection is shown in Figure 4 for the value K = 2 for which $\lambda_K = 3\pm 2\sqrt{2}$ giving values of (.344, .172) and (11.656, 5.828) as the two points of intersection shown. The behavior of the solutions changes as μ varies along a curve $\mu = K\lambda$. For small values of μ, a Case 3 type solution occurs. As μ increased the first

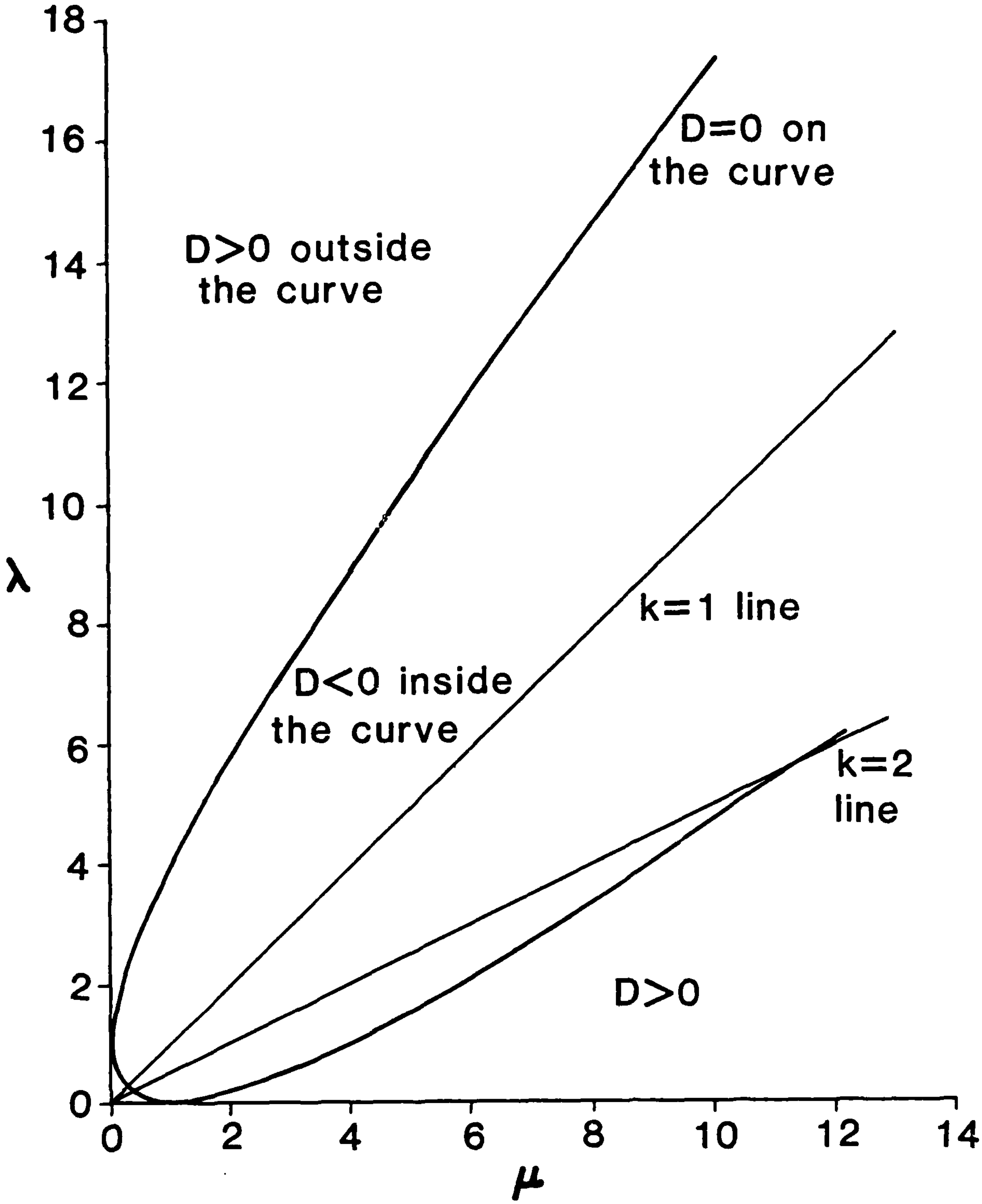

Figure 4

Regions of the (μ,λ) Plane

intersection point is reached, where $D = 0$, giving a Case 2 solution. Further increase in μ moves the solution through a range where $D < 0$ and a Case 1 type solution governs the model. For $K = 1$, this situation continues. For all other values of K, increase in μ takes the solution to the second intersection, a Case 2 form, then on to Case 3 form solutions again. The regions specified in Figure 4 can be helpful in determining the general form of solutions to be expected when using various (μ,λ) combinations and to see how the solutions will change as the (μ,λ) values change.

The roots of $Q(s)$ are given by: $-\alpha\pm\sqrt{\frac{D}{4}}$ where $\alpha = (1+\lambda+\mu)/2$. Let $\beta^2 = D/4$. Then the values of D as a function of μ and λ yield three distinct forms of solution as indicated by the three regions of the (μ,λ) plane shown in Figure 3. These cases are defined by: $D < 0$ gives Case 1, $D = 0$ gives Case 2, and $D > 0$ gives Case 3.

In Case 1 the quadratic $Q(s) = (s+\alpha)^2 + \beta^2$; in Case 2, $Q(s) = (s+\alpha)^2$; and in Case 3, $Q(s) = (s+\alpha)^2 - \beta^2$. The form of the roots of $Q(s)$ in terms of β can be treated in various ways and care must be exercised to get the correct form of the solution in each of the cases and also in numerical calculations as discussed in Section 4.

Section 2.2 Case 1 Solutions

In this case, $D < 0$ and $Q(s)$ has complex roots. The quantity β^2 is negative. It is useful to represent $Q(s)$ as a quadratic of the form $Q(s) = (s+\alpha)^2 - \beta^2$. Then F_s can be expressed as follows:

$$F_s = s_1A(s) + w_1B(s) + a_1C(s), \quad \text{where}$$

$$A(s) = \frac{s^2 + (1+\lambda)s + \lambda}{s[(s+\alpha)^2 - \beta^2]} = \frac{A_1}{s} + \frac{B_1s + C_1}{(s+\alpha)^2 - \beta^2}.$$

Direct calculation yields:

$$A_1 = \frac{\lambda}{\lambda+\mu+\lambda\mu}, \quad B_1 = \frac{\mu+\lambda\mu}{\lambda+\mu+\lambda\mu}, \quad C_1 = \frac{\mu+\lambda\mu+\lambda^2\mu}{\lambda+\mu+\lambda\mu}.$$

In some formulas, it is convenient to express $\lambda+\mu+\lambda\mu$ as $\alpha^2 - \frac{D}{4}$ which for Case 1 gives $\alpha^2 - \beta^2$ since $\beta^2 = D/4$. However, it must be observed that in this case, $\alpha^2 - \beta^2$ is in fact the sum of two positive quantities in order to get the correct inverse Laplace transform expression. Proceeding, we find

$$B(s) = \frac{s + \lambda}{s[(s+\alpha)^2 - \beta^2]} = \frac{A_2}{s} + \frac{B_2 s + C_2}{(s+\alpha)^2 - \beta^1},$$

where

$$A_2 = A_1 = \frac{\lambda}{\lambda+\mu+\lambda\mu}, \quad B_2 = \frac{-\lambda}{\lambda+\mu+\lambda\mu}, \quad C_2 = \frac{\mu-\lambda^2}{\lambda+\mu+\lambda\mu}.$$

$$C(s) = \frac{\lambda}{s[(s+\alpha)^2 - \beta^2]} = \frac{A_3}{s} + \frac{B_3 s + C_3}{(s+\alpha)^2 - \beta^2},$$

where $A_3 = A_1$, $B_3 = B_2$, $C_3 = \dfrac{-\lambda(a+\lambda+\mu)}{\lambda+\mu+\lambda\mu}$.

The inverse transforms are:

$$L^{-1}[A(s)] = A_1 + B_1 e^{-\alpha t}\cos\beta t + \frac{(C_1 - B_1\alpha)}{\beta} e^{-\alpha t}\sin\beta t,$$

$$L^{-1}[B(s)] = A_1 + B_2 e^{-\alpha t}\cos\beta t + \frac{(C_2 - B_2\alpha)}{\beta} e^{-\alpha t}\sin\beta t,$$

$$L^{-1}[C(s)] = A_1 + B_2 e^{-\alpha t}\cos\beta t + \frac{(C_3 - B_2\alpha)}{\beta} e^{-\alpha t}\sin\beta t,$$

Thus: $m_s(t) = s_1 L^{-1}[A(s)] + w_1 L^{-1}[B(s)] + a_1 L^{-1}[C(s)]$.

Next, the quantity $m_a(t)$ is developed. Its transform is given by:

$$F_a = \frac{a_1}{s+\lambda} + \frac{\mu}{s+\lambda} F_s$$

$$F_a = \frac{a_1}{s+\lambda} + \frac{s_1\mu}{(s+\lambda)^2-\beta^2} + \frac{s_1\mu + w_1\mu}{s[s+\alpha)^2-\beta^2]} + \frac{a_1\lambda\mu}{(s+\lambda)s[(s+\alpha)^2-\beta^2]} .$$

After partial fraction expansion and algebraic arrangement, this quantity becomes:

$$F_a = \frac{\mu(s_1+w_1+a_1)}{\alpha^2 - \beta^2}\frac{1}{s}$$

$$+ \frac{[s_1\mu(\lambda+\mu+2\lambda\mu-1) - w_1\mu(1+\lambda+\mu) + a_1\lambda(1+\mu^2-\lambda-\lambda\mu)]}{2(\alpha^2-\beta^2)[(s+\alpha)^2 - \beta^2]}$$

$$+ \frac{(s+\alpha)[-(s_1+w_1)\mu + a_1\lambda(1+\mu)]}{(\alpha^2-\beta^2)[(s+\alpha)^2 - \beta^2]}$$

Inversion of this transform yields $m_a(t)$. Continuing, the transform of $m_w(t)$ is:

$$F_w = (s+\mu)F_s - s_1,$$

$$F_w = \frac{\mu\lambda(s_1+w_1+a_1)}{s[(s+\alpha)^2-\beta^2]} + \frac{w_1(s+\alpha)}{(s+\alpha)^2 - \beta^2} + \frac{w_1(\lambda+\mu) + a_1\lambda - w_1\alpha}{(s+\alpha)^2 - \beta^2},$$

After some algebraic arrangement, this can be written:

$$F_w = \frac{\mu\lambda(s_1+w_1+a_1)}{\lambda+\mu+\lambda\mu}\frac{1}{s} + \frac{s+\alpha}{(s+\alpha)^2 - \beta^2}\,\frac{w_1(\lambda+\mu) - \mu\lambda(s_1+a_1)}{\lambda+\mu+\lambda\mu}$$

$$+ \frac{[w_1(\lambda^2+\mu^2-\lambda-\mu) + a_1\lambda(2\lambda+\mu+\lambda\mu-\mu^2) - s_1\lambda\mu(1+\mu+\lambda)]}{2(\alpha^2-\beta^2)[(s+\alpha)^2 - \beta^2]} .$$

Inversion of this transform yields $m_w(t)$. The Case 1 solutions, obtained by means of Laplace transforms as shown above, are:

$$m_s(t) = \lambda A + \frac{1}{\alpha^2-\beta^2}\,[\mu s_1(1+\lambda) - \lambda(a_1+w_1)]e^{-\alpha t}\cos\beta t$$

$$+ \frac{1}{2\beta(\alpha^2-\beta^2)}\,[\mu s_1(1+\lambda^2-\mu-\mu\lambda) + w_1(2\mu-\lambda^2+\lambda+\lambda\mu)$$

$$- \lambda a_1(1+\lambda+\mu)]e^{-\alpha t}\sin\beta t$$

$$m_a(t) = \mu A + \frac{1}{\alpha^2-\beta^2}\,[-(s_1+w_1)\mu + a_1\lambda(1+\mu)]e^{-\alpha t}\cos\beta t$$

$$+ \frac{1}{2\beta(\alpha^2-\beta^2)}\,[s_1\mu(\lambda+\mu+2\lambda\mu-1) - w_1\mu(1+\lambda+\mu)$$

$$+ a_1\lambda(1+\mu^2-\lambda-\lambda\mu)]e^{-\alpha t}\sin\beta t$$

$$m_w(t) = \mu\lambda A + \frac{1}{\alpha^2-\beta^2}\,[w_1(\lambda+\mu) - \lambda\mu(s_1+a_1)]e^{-\alpha t}\cos\beta t$$

$$+ \frac{1}{2\beta(\alpha^2-\beta^2)}\,[w_1(\lambda^2+\mu^2-\lambda-\mu) + a_1\lambda(2\lambda+\mu+\lambda\mu-\mu^2)$$

$$- s_1\lambda\mu(1+\mu+\lambda)]e^{-\alpha t}\sin\beta t$$

In these equations, $A = A_1$.

As a check on these rather involved results, it is true that $m_a(t) + m_w(t) + m_s(t) = a_1 + w_1 + s_1$ as required by the conservation of units assumption in the model formulation. This direct though somewhat tedious calculation is omitted from this report. Some additional checks on the calculations will be discussed at the end of the chapter.

Section 2.3 Case 2 Solutions

In Case 2, D = 0 so that Q(s) has two real roots both equal to $-\alpha$. Thus $Q(s) = (s+\alpha)^2$ so that the transform of $m_s(t)$ can be written as:

$$F_s = \frac{s_1 s^2 + [s_1(1+\lambda) + w_1]s + (s_1+w_1+a_1)\lambda}{s(s+\alpha)^2},$$

$$= A(s) + (s_1+w_1+a_1)\lambda B(s),$$

where $A(s) = \dfrac{s_1 s + s_1(1+\lambda) + w_1}{(s+\alpha)^2}$, $B(s) = \dfrac{1}{s(s+\alpha)^2}$.

Partial fraction expansion yields:

$$A(s) = \frac{w_1 + s_1(1+\lambda-\alpha)}{(s+\alpha)^2} + \frac{s_1}{s+\alpha},$$

$$B(s) = \frac{1}{\alpha^2 s} - \frac{1}{\alpha(s+\alpha)^2} - \frac{1}{\alpha^2(s+\alpha)},$$ so that:

$$F_s = \frac{(s_1+w_1+a_1)\lambda}{\alpha^2}\frac{1}{s} + [s_1 - \frac{(s_1+w_1+a_1)\lambda}{\alpha^2}]\frac{1}{s+\alpha}$$

$$+ [w_1 + s_1(1+\lambda-\alpha) - \frac{(s_1+w_1+a_1)\lambda}{\alpha}]\frac{1}{(s+\alpha)^2}.$$

The transform of $m_w(t)$ is given by $F_w = (s+\mu)F_s - s_1$. From the above expression for F_s, we can write

$$F_s = \frac{R_1}{s} + R_2\frac{1}{s+\alpha} + R_3\frac{1}{(s+\alpha)^2}$$

where $R_1 = \dfrac{(s_1+w_1+a_1)\lambda}{\alpha^2}$, $r_2 = s_1 - R_1$,

and $R_3 = w_1 + s_1(1+\lambda-\alpha) - \alpha R_1$.

Then F_w can be written

$$F_w = R_1 \frac{(s+\mu)}{s} + R_2 \frac{(s+\mu)}{s+\alpha} + R_3 \frac{(s+\mu)}{(s+\alpha)^2} - s_1$$

Partial fraction expansion and simplification yields:

$$F_w = R_1\mu \frac{1}{s} + [R_2(\mu-\alpha)+R_3]\frac{1}{s+\alpha} + R_3(\mu-\alpha)\frac{1}{(s+\alpha)^2}.$$

The transform of $m_a(t)$ is given by:

$$F_a = \frac{a_1}{s+\lambda} + \mu R_1 A(s) + \mu R_2 B(s) + \mu R_3 C(s), \qquad \text{where}$$

$$A(s) = \frac{1}{s(s+\lambda)}, \qquad B(s) = \frac{1}{(s+\lambda)(s+\alpha)}, \quad \text{and} \quad C(s) = \frac{1}{(s+\lambda)(s+\alpha)^2}.$$

Partial fraction expansion yields:

$$A(s) = \frac{1}{\lambda s} - \frac{1}{\lambda(s+\lambda)}, \qquad B(s) = \frac{1}{(\alpha-\lambda)(s+\lambda)} - \frac{1}{(\alpha-\lambda)(s+\alpha)},$$

$$\text{and} \quad C(s) = \frac{1}{(\alpha-\lambda)^2(s+\lambda)} - \frac{1}{(\alpha-\lambda)(s+\alpha)^2} - \frac{1}{(\alpha-\lambda)^2(s+\alpha)}.$$

These quantities give F_a in the following form:

$$F_a = \frac{\mu R_1}{\lambda}\frac{1}{s} + [a_1 - \frac{\mu R_1}{\lambda} + \frac{\mu R_2}{\alpha-\lambda} + \frac{\mu R_3}{(\alpha-\lambda)^2}]\frac{1}{s+\lambda}$$

$$+ [-\frac{\mu R_2}{\alpha-\lambda} - \frac{\mu R_3}{(\alpha-\lambda)^2}]\frac{1}{s+\alpha} - \frac{\mu R_3}{\alpha-\lambda}\frac{1}{(s+\alpha)^2}$$

The coefficient of $\frac{1}{s+\lambda}$ is equal to zero as can be established by direct algebraic calculation.

Inversion of the transforms developed above yields the Case 2 solutions. To simplify the notation in these solutions, let $B = (s_1+w_1+a_1)/\alpha^2$, and $C = w_1+s_1(1+\lambda-\alpha) - \lambda\alpha B$. Then the Case 2 solutions are:

$$m_s(t) = \lambda B + [s_1-\lambda B]e^{-\alpha t} + Cte^{-\alpha t}$$

$$m_a(t) = \mu B + \frac{\mu}{(\lambda-\alpha)^2}[-s_1-w_1+\lambda(2\alpha-\lambda)B]e^{-\alpha t} - \frac{\mu}{\alpha-\mu}Cte^{-\alpha t}$$

$$m_w(t) = \mu\lambda B + [w_1 - \lambda\mu B]e^{-\alpha t} + (\mu-\alpha)Cte^{-\alpha t}$$

By direct calculation, it can be shown that the conservation assumption is satisfied for these solution functions, i.e., that $m_a(t)+m_w(t)+m_s(t) = a_1+w_1+s_1$ for all values of t.

Section 2.4 Case 3 Solution

In this case, $D > 0$ and $Q(s)$ has two distinct real roots: $\theta_1 = -\alpha + \frac{\sqrt{D}}{2}$, $\theta_2 = -\alpha - \frac{\sqrt{D}}{2}$. It is convenient to sometimes express $\frac{\sqrt{D}}{2}$ as β in the solution forms for Case 3. In this notation, the roots of $Q(s)$ are $-\alpha+\beta$, and $-\alpha-\beta$.

The transform of $m_s(t)$ in Case 3 has the form:

$$F_s = \frac{s_1(s+1)(s+\lambda)}{s(s-\theta_1)(s-\theta_2)} + \frac{w_1(s+\lambda)}{s(s-\theta_1)(s-\theta_2)} + \frac{a_1\lambda}{s(s-\theta_1)(s-\theta_2)}$$

$$= A(s) + (a_1+w_1+s_1)\lambda B(s),$$

where $$A(s) = \frac{s_1 s + s_1(1+\lambda) + w_1}{(s-\theta_1)(s-\theta_2)}, \qquad B(s) = \frac{1}{s(s-\theta_1)(s-\theta_2)}$$

Expansion gives:

$$A(s) = \frac{A_1}{s-\theta_1} + \frac{B_1}{s-\theta_2},$$

where $$A_1 = \frac{s_1\theta_1 + s_1(1+\lambda) + w_1}{\theta_1-\theta_2},$$

and $$B_1 = -\frac{s_1\theta_2 + s_1(1+\lambda) + w_1}{\theta_1-\theta_2}.$$

$$B(s) = \frac{A_2}{s} + \frac{B_2}{s-\theta_1} + \frac{C}{s-\theta_2}, \qquad \text{where } A_2 = \frac{1}{\theta_1\theta_2},$$

$$B_2 = \frac{1}{\theta_1(\theta_1-\theta_2)} \qquad \text{and} \qquad C = \frac{-1}{\theta_2(\theta_1-\theta_2)}.$$

These expressions result in the value:

$$F_s = H_1\frac{1}{s} + H_2\frac{1}{s-\theta_1} + H_3\frac{1}{s-\theta_2},$$

where

$$H_1 = \frac{(a_1+w_1+s_1)\lambda}{\lambda+\mu+\lambda\mu} \qquad H_2 = \frac{s_1\theta_1+s_1(1+\lambda)+w_1}{2\beta} + \frac{(a_1+w_1+s_1)\lambda}{2\beta\theta_1},$$

and $$H_3 = \frac{-s_1\theta_2 - s_1(1+\lambda) - w_1}{2\beta} - \frac{(a_1+w_1+s_1)\lambda}{2\beta\theta_2}$$

The transform of $m_w(t)$ has the form:

$$F_w = (s+\mu)F_s - s_1.$$

Direct calculation shows that $H_1+H_2+H_3 = S_1$ giving the following form for F_w:

$$F_w = H_1\mu \frac{1}{s} + H_2(\theta_1+\mu)\frac{1}{s-\theta_1} + H_3(\theta_2+\mu)\frac{1}{s-\theta_2} .$$

The transform of $m_a(t)$ has the form:

$F_a = \frac{a_1}{s+\lambda} + \frac{\mu}{s+\lambda} F_s$, which can be put into the form:

$$F_a = \frac{H_1\mu}{\lambda}\frac{1}{s} + \frac{H_2\mu}{\lambda+\theta_1}\frac{1}{s-\theta_1} + \frac{H_3\mu}{\lambda+\theta_2}\frac{1}{s-\theta_2}$$

$$+ \left[a_1 - \frac{H_1\mu}{\lambda} - \frac{H_2\mu}{\lambda+\theta_1} - \frac{H_3\mu}{\lambda+\theta_2}\right]\frac{1}{s+\lambda} .$$

It can be shown that the coefficient of $\frac{1}{s+\lambda}$ is equal to zero.

Inversion of the transforms given above yields the Case 3 solutions in the following form.

$$m_s(t) = H_1 + H_2e^{\theta_1 t} + H_3e^{\theta_2 t}$$

$$m_w(t) = \mu H_1 + H_2(\theta_1+\mu)e^{\theta_1 t} + H_3(\theta_2+\mu)e^{\theta_2 t}$$

$$m_a(t) = \frac{H_1\mu}{\lambda} + \frac{H_2\mu}{\lambda+\theta_1}e^{\theta_1 t} + \frac{H_3\mu}{\lambda+\theta_2}e^{\theta_2 t}$$

The results given in this Chapter are rather complicated, and it is desirable to check their correctness before using them in numerical analyses such as those given in Chapter 4. Two types of tests on solution correctness have been made. For each case it has been shown by direct calculation that $m_a(t)+m_w(t)+m_s(t) = a_1+w_1+s_1$ as required by the conservation assumption. A more sophisticated test of solution correctness is obtained by noting that both Case 1 and Case 3 solutions should tend to the Case 2 solutions as $\beta \to 0$. By allowing β to approach zero in the Case 1 and Case 3 solutions, it is found that the Case 2

solutions are indeed obtained. Though the limit calculations are detailed to carry out, they are direct calculations and are omitted from this report.

It should be noted that the initial value of time t_o has been taken as $t_o=0$ in deriving the above forms of solution. This means that in model analyses, the time variable must be redefined each time the model changes region form. This is discussed more fully in Chapter 4.

CHAPTER 3. REGION 2 SOLUTIONS

As specified in Section 1.2, the Region 2 form of the bulk availability model is given by the value $K(t) = r + (\mu-1)m_s(t)$. This results in the following system of differential equations:

$$\dot{m}_a(t) = \mu m_s(t) - \lambda m_a(t),$$

$$\dot{m}_w(t) = \lambda m_a(t) - r - (\mu-1)m_s(t),$$

$$\dot{m}_s(t) = r - m_s(t).$$

Initial conditions are specified for some value of time, t_o, at which the model behavior becomes governed by the Region 2 form. As in Chapter 2, the initial values are defined to be $m_a(t_o) = a_1$, $m_s(t_o) = s_1$, and $m_w(t_o) = w_1$. Because of the conservation assumption, the Region 2 solutions also satisfy the condition:

$$m_a(t) + m_s(t) + m_w(t) = a_1 + s_1 + w_1 \text{ for all } t.$$

The equation for $m_s(t)$ is not coupled to the quantities $m_a(t)$ and $m_w(t)$ so that it can be solved directly for $m_s(t)$. It is a first order linear differential equation whose solution is:

$$m_s(t) = s_1 e^{-t} + r(1-e^{-t}).$$

This value can be put into the equation for $m_a(t)$ to obtain the equation:

$$\dot{m}_a(t) = \mu r + \mu(s_1-r)e^{-t} - \lambda m_a(t).$$

This equation is solved directly to obtain:

$$m_a(t) = \frac{\mu r}{\lambda}(1-e^{-\lambda t}) + \frac{\mu(s_1-r)}{\lambda-1}(e^{-t} - e^{-\lambda t}) + a_1 e^{-\lambda t}.$$

Now using the values already obtained for $m_s(t)$ and $m_a(t)$, the equation for $m_w(t)$ becomes:

$$\dot{m}_w(t) = [\frac{\mu r-\lambda\mu s_1}{\lambda-1} + \lambda a_1]e^{-\lambda t} + \frac{(s_1-r)}{\lambda-1}(\lambda+\mu-1)\ e^{-t}.$$

Direct integration of this differential equation yields:

$$m_w(t) = a_1+w_1+s_1 - \frac{\mu r}{\lambda} - r - \frac{1}{\lambda}[\frac{\mu r-\lambda\mu s_1}{\lambda-1} + \lambda a_1]e^{-\lambda t}$$

$$- \frac{(s_1-r)(\lambda+\mu-1)}{\lambda-1} e^{-t}.$$

As a check on these results, direct calculation shows that the conservation condition holds for the Region 2 solutions given above.

It should be noted that the initial value of time t_0 has been taken as t_0=0 in deriving the above forms of solution. This means that in model analyses, the time variable must be redefined each time the model changes region form. This is discussed more fully in Chapter 4.

Because of the simple form in which the Region 2 solutions depend on t, the limit values as t increases are easily obtained. They are:

$$\lim_{t\to\infty} m_s(t) = r,$$

$$\lim_{t\to\infty} m_a(t) = \frac{\mu r}{\lambda}$$

$$\lim_{t\to\infty} m_w(t) = a_1+w_1+s_1 - \frac{\mu r}{\lambda} - r.$$

It should be observed however that as t increases, the model may not remain in Region 2. If it does not, then the limit values are undefined. The condition for the model to be in Region 2 is that $m_w(t) \geqq r + (\mu-1)m_s(t)$. The limit form of this condition requires that $a_1+w_1+s_1 \geqq \mu r + \frac{\mu r}{\lambda} + r$.

This is a condition that relates all six parameters of the model to give a set of values that result in the Region 2 limit forms for the model descriptors m_a, m_s, and m_w.

Though numerical evaluation is required for typical model analyses in either Region 1 or Region 2, some special cases can be considered directly. As an illustration of such a consideration, the following special case has been studied.

Consider the Region 2 solution in which $s_1 = r$ so that the service facility starts fully utilized. Then $m_s(t) = r$ and the facility continues to be fully utilized. In this case, the remaining units are distributed between the available and waiting states according to the functions:

$$m_a(t) = \frac{\mu r}{\lambda} + (a_1 - \frac{\mu r}{\lambda})e^{-\lambda t},$$

$$m_w(t) = a_1 + w_1 - \frac{\mu r}{\lambda} - (a_1 - \frac{\mu r}{\lambda})\, e^{-\lambda t}.$$

The condition for this to govern the distribution of units is that $m_w(t) \geqq \mu r$; if this condition is not satisfied, the system will move into the Region 1 solution. In particular, for a steady state limit solution to exist in this form, it is required that $a_1+w_1 \geqq \mu r(1+\frac{1}{\lambda})$. It is an indication of the intrinsic non-linearity of the model that the steady state depends on initial values. An example illustrating the situation when the condition is satisfied is given in Section 4.2.

CHAPTER 4. NUMERICAL EXAMPLES OF SOLUTIONS

The bulk availability model has been formulated so that its descriptive functions $m_a(t)$, $m_s(t)$, and $m_w(t)$ have different forms in the solution Regions 1 and 2. As the system moves from one region of solution to another, time must be reinitialized and the initial distribution of units among the three states of the model must be specified by the values obtained in the previous region. The solutions themselves have been obtained in explicit form but are rather complicated. Certainly they are too involved for hand calculation. In this chapter, a computer implementation is described, and it is used to obtain some example results. These examples illustrate the use of the model for bulk availability analyses and show the kind of results obtained from the mathematical solutions. A major feature of interest in model analysis is the relation of the system descriptors $m_a(t)$, $m_s(t)$, and $m_w(t)$ to the system parameters λ, μ, r, and model parameters a_1, s_1, and w_1.

Section 4.1 Computerized Solution Calculation

As pointed out in each Region solution section, the solution forms given are based on an assumption that the initial time value is zero. In

a real solution where the model form may pass from one solution form to another, it is necessary to use two time values. One value of time represents the actual time while the other time value is the time within a particular region solution. Each change from one region form to another requires an initialization of the "local" time variable to zero. When this is done, the actual time variable is incremented by a value equal to the local time value just prior to its initialization and the change to a new region form solution. Depending on the six parameters that define the model, three types of solutions can arise:

(i) the solution can start in one region and remain in that region for the duration of the analysis;

(ii) the solution can start in one region and move to the other which then determines its value for the duration of the analysis;

(iii) the solution can move from one region to another then back again, continuing to change regions.

A combination of (ii) and (iii) can also occur, but the major types of behavior are given by these forms of solution. Thus a computerized model analysis must be formulated in such a way that it can represent each of these distinct kinds of system behavior.

In addition to keeping track of actual time, it is necessary to match the values of $m_a(t)$, $m_s(t)$, and $m_w(t)$ at the end of one model form situation with the initial values of these quantities at the next model form when a change of form is required.

Figure 5 shows a flow chart of how the computerized model analysis is carried out. To keep track of time, three different "time like" variables are used. The variable T measures time from the start of one Region form. It changes in increments of specified amount DT.

The value of DT must be selected rather carefully because it governs the time values at which the solution forms are evaluated. If the solution remains in a single Region, DT can be relatively large, its value subject only to the detail of solution form required in a particular analysis. However, when the solution changes Regions, it should do so as near to the critical values of the solution as possible. If DT is too large, the solution from one Region will continue into the other Region on transition and the "initial values" will be incorrect being based on the previous solution form which has ceased to be appropriate at some previous time. This effect can introduce significant errors in a model analysis. The most appropriate values for DT in a specific analysis can be determined by a try-and-test procedure. Fortunately, a computer run of the model does not require extensive time or storage so it is feasible to try several values of DT to insure a realistic set of results. The variable TOTT records each of the times spent by the system in each solution Region. Each time there is a change of region, the value of TOTT up to the start of that Region solution is increased by the final value of T for that solution. Actual time is given by the variable TLOC which is the sum of the present value of TOTT and the value of T obtained so far in the present Region solution at the termination of the analysis. An analysis is terminated when TLOC reaches a value TF specified by the program. The quantities DT and TF are input values that govern the detailed calculation of the solutions, they are programming parameters rather than system parameters.

The solution Region that should be used is determined by the condition $m_w(t) < r-m_s(t)+\mu m_s(t)$ as discussed in Section 1.2 (Figure 2). The proper Region is designated by a flag variable KR which takes the values 1 and 2 to denote Regions 1 and 2 respectively.

At each iteration of the solution evaluation procedure, the condition on $m_w(t)$ is checked to see if a change of Region should take place. When a change of Region takes place TOTT is updated, the values of $m_a(T)$, $m_s(T)$, and $m_w(T)$ are used as the intial values for the next Region solution form, and T is initialized to DT as a starting value for the new Region solution. This procedure is indicated in Figure 5. In the detailed program, a listing of which is given in the Appendix, care must be taken in evaluating the Region 1 solutions. In that region, the three Cases give different forms of solutions as described in Chapter 2. However, the notation defining the quantity β must be properly interpreted in each case. This is particularly true in Case 3 where $D < 0$ so that β^2 is in fact negative making $\alpha^2-\beta^2$ a positive quantity for all values of α and β. The detailed program must test for each case and interpret the numerical signs so as to give the correct results. It must also determine how often solution values are to be given as output and what detail of output information is required. These features are indicated by additional program parameters as described in the Appendix.

In the next section, computerized analyses of several examples are given. They were obtained using the program given in the Appendix which follows the flow diagram of Figure 5. Examples such as these require very short running times and little storage for their execution.

Section 4.2 <u>Examples</u>

To illustrate the kinds of solutions that can be obtained from the model, some examples are given in this section. First, a simple example is given for which the solution form can be expressed explicitly. Then

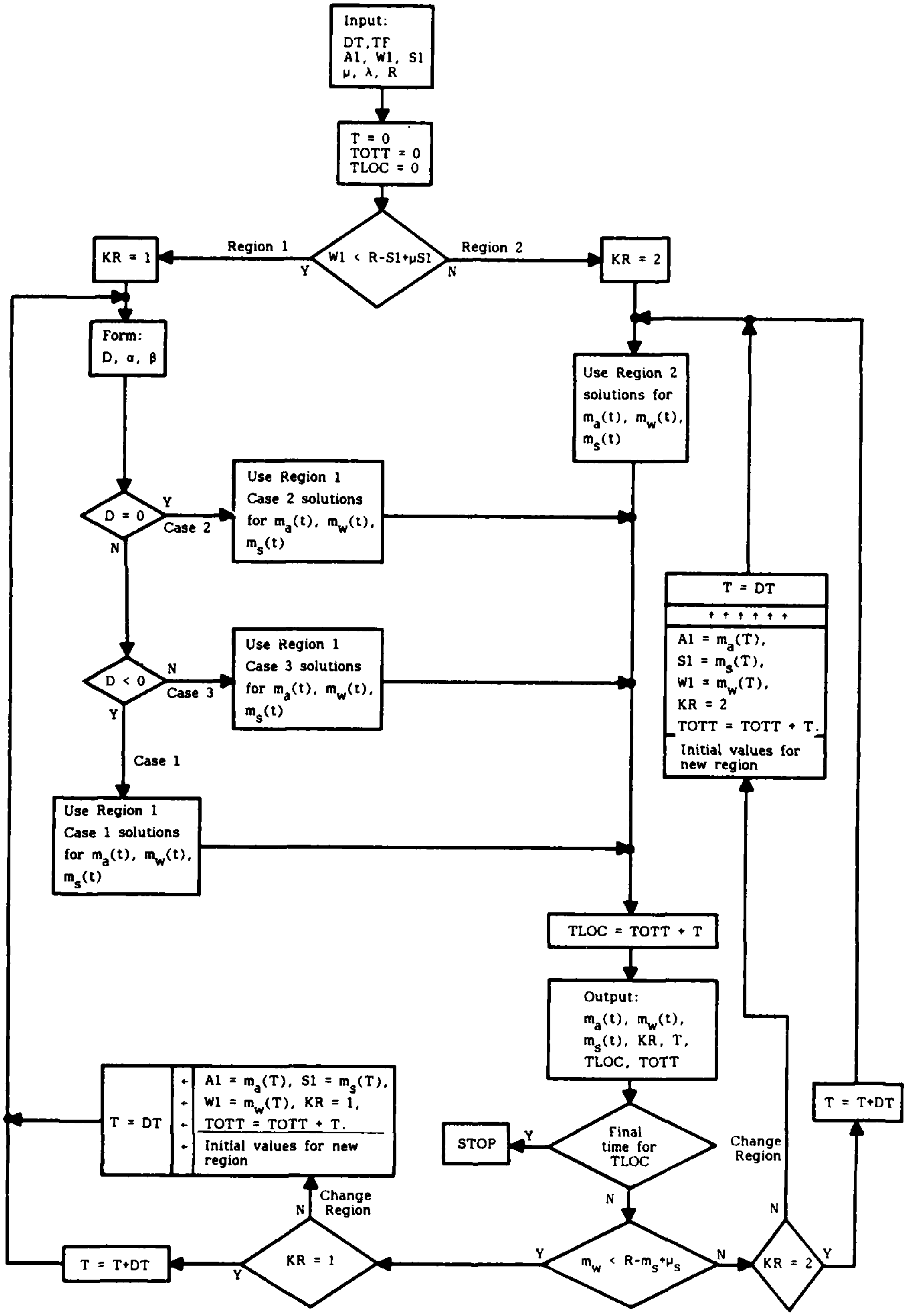

Figure 5

Flow Diagram of Computational Program

some numerical examples are given that employ the computer evaluation procedure described above in Section 4.1.

Example 1. Let $\lambda = 1$, $\mu = 2$, $r = 10$, and assume initial values $a_1 = 70$, $s_1 = 10$, and $w_1 = 20$. The constant value of total population of units is therefore 100. The condition: $w_1 < r-s_1+\mu s_1$ becomes $20 < 10-10+20$ which is not satisfied so a Region 2 type of solution is obtained. In Chapter 3, the limit behavior of the model when $s_1 = r$ was discussed and it was found that the solution would remain in Region 2 provided the condition $a_1+w_1 \geqq \mu r(1+\frac{1}{\lambda})$ was satisfied. In the present example, this condition has the values $70+20 \geqq 10(1+1)$ and is satisfied. Therefore, the solution starts in Region 2 and remains in Region 2. The explicit solution is:

$$m_a(t) = 20 + 50e^{-t}$$
$$m_w(t) = 70 - 50e^{-t}$$
$$m_s(t) = 10$$

The total number of units is maintained at 100 and, in the steady state, 20 are available, 10 are in service, and 70 are waiting for service.

A modification of the initial conditions for this example yields quite different results. Let $a_1 = 80$, $s_1 = 10$, and $w_1 = 10$. Then the condition $w_1 < r-s_1+\mu s_1$ is satisfied in the form $10 < 20$ and the solution is in Region 1 with $D = -4$, $\beta^2 = -1$, and $\alpha = 2$. Thus Case 1 type solutions must be used to represent the descriptive functions, which have the explicit form:

$$m_s(t) = 20 - 10e^{-2t} \cos t - 30e^{-2t} \sin t$$
$$m_a(t) = 40 + 40e^{-2t} \cos t + 20e^{-2t} \sin t.$$
$$m_w(t) = 40 - 30e^{-2t} \cos t + 10e^{-2t} \sin t.$$

The conservation condition $m_a(t)+m_s(t)+m_w(t) = 100$ holds, as it should, for these solutions.

The Case 1 examples are interesting because the D < 0 condition permits oscillations to occur causing the level of availability, for example, to move up and down about a steady state value. This is illustrated by putting the availability function $m_a(t)$ into the following form:

$$m_a(t) = 40 + 20\sqrt{5}\, e^{-2t} \sin(t+\theta)$$

where $\sin\theta = \frac{2}{\sqrt{5}}$. In numerical terms, the function is expressed as

$$m_a(t) = 40 + 44.72e^{-2t} \sin(t+1.1).$$

The limit value is 40 and the values of $m_a(t)$ undergo damped oscillation about this value as shown in Figure 6. The limit values of $m_s(t)$ and $m_w(t)$ are seen to be equal to 20 and 40 respectively.

The remaining examples of this section are given in graphical form where the function values are obtained by using the computer procedure indicated in Figure 5. Numerical values for each example are given in the Appendix.

<u>Example 2</u>. To illustrate a Region 1 solution, let $\mu = .4$, $\lambda = .01$, $r = 10$, and set the initial values equal to $a_1 = 89$, $s_1 = 5$, $w_1 = 6$ so that the total number of units is 100. For this example, $D = .33 > 0$ so it has a Case 3 solution. The values of $m_a(t)$, $m_s(t)$, and $m_w(t)$ are shown in Figure 7 for 20 time units. Because the solution does not change regions, the time increment used in the evaluation program can be large. A value of DT = 1 was used for the numerical values shown in Figure 7.

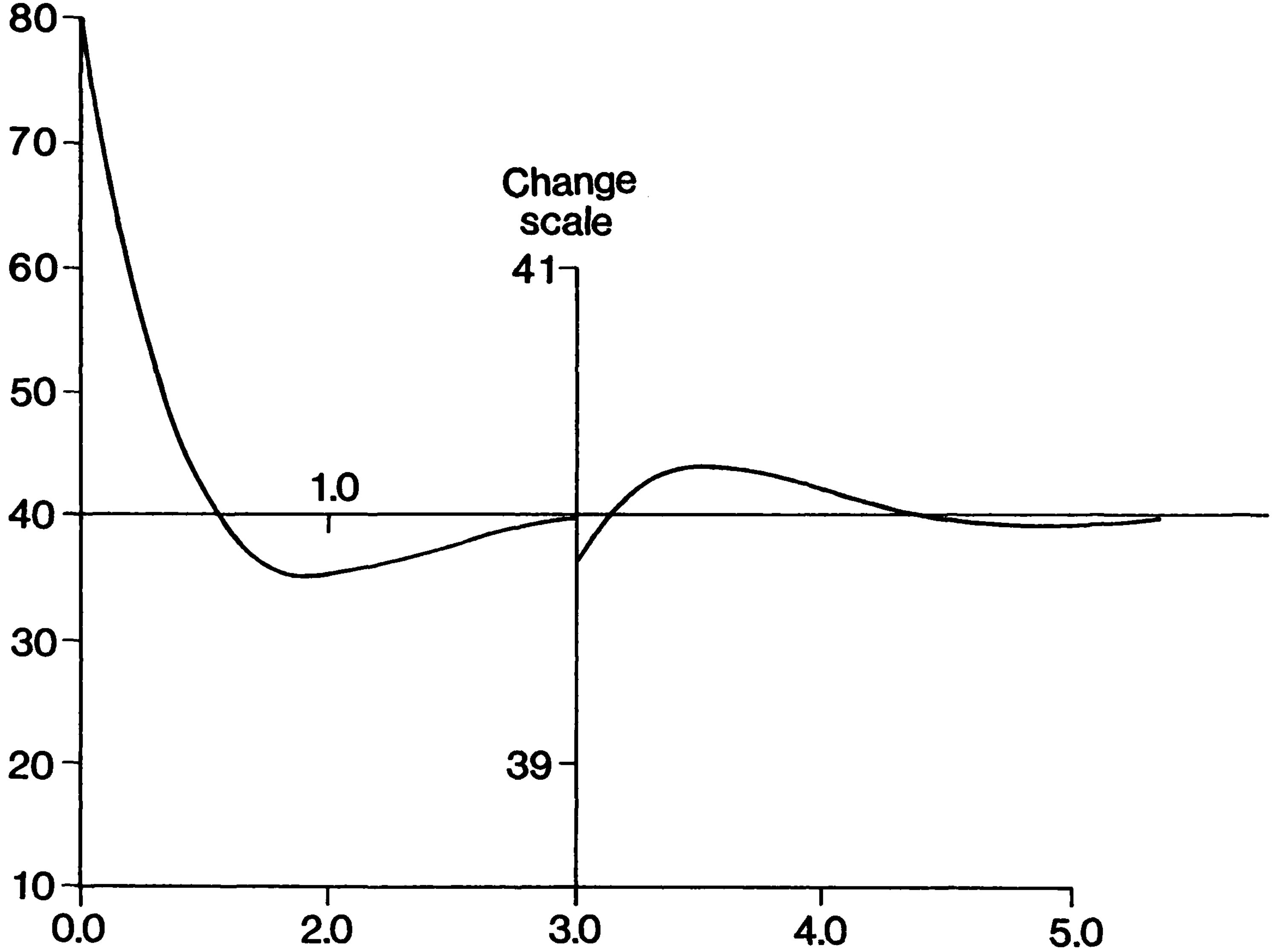

Figure 6

Example 1

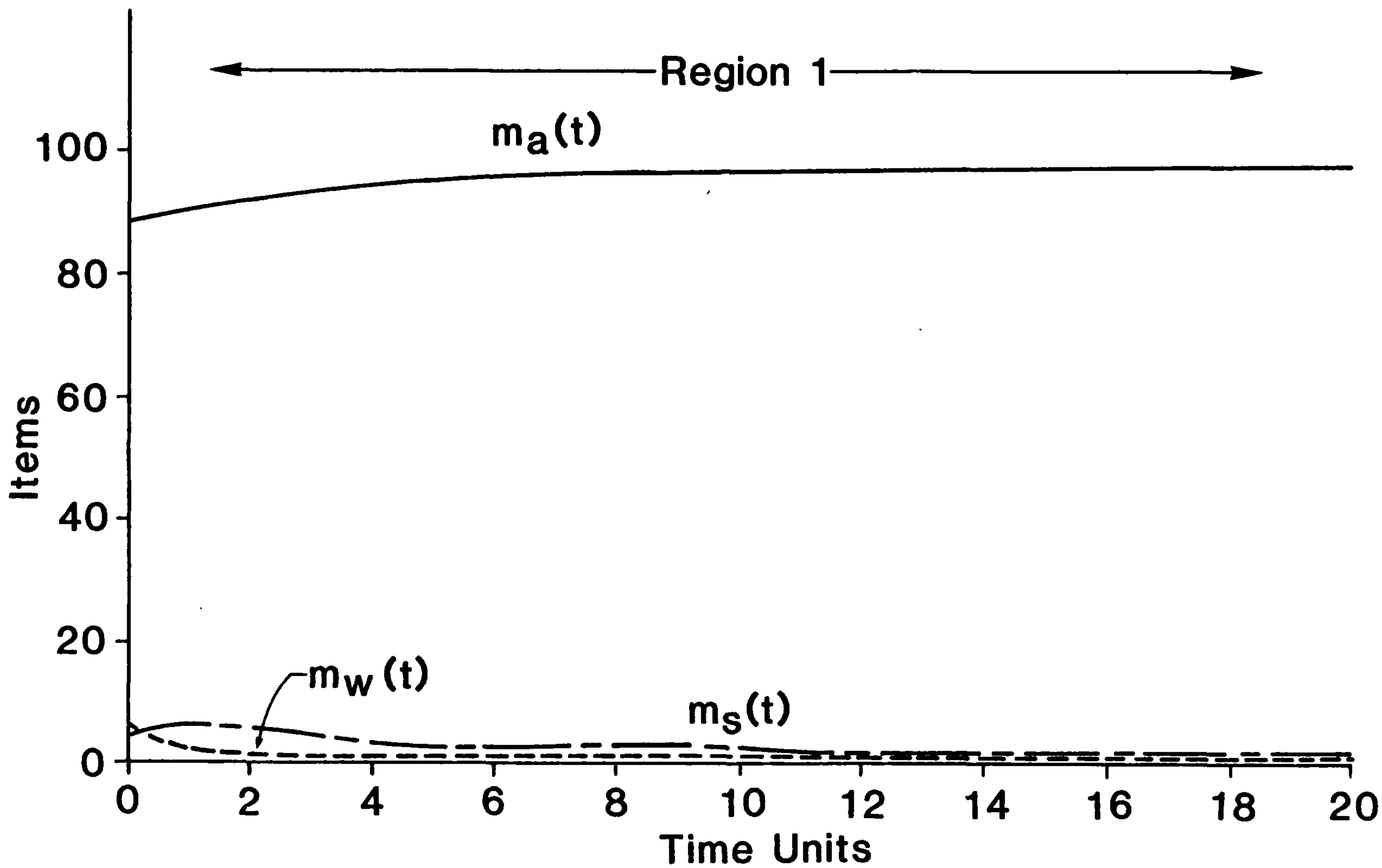

Figure 7

Example 2

Bulk availability of the kind shown in this example can be characterized as a rather pure or classical type of availability behavior as distinct from a classical reliability or mixed forms illustrated in other examples. The service rate is greater than the failure rate by a large enough amount that the availability function is able to reach a steady state value of approximately 97 units. Of the remaining three units, two are in service and one is waiting for service, in the steady state.

Example 3. A Region 2 solution is illustrated in this example. The parameter values are: $\mu = .2$, $\lambda = .2$, and $r = 10$. Initial values are taken to be $a_1 = 70$, $s_1 = 10$, and $w_1 = 20$ giving a total of 100 units. Here, the initial number in service, s_1, is equal to the capacity of the service facility. Under these conditions with equal values for service and failure rates, the larger number of units initially available forces the size of the population of units waiting for service to increase. This continues until a steady state is reached with $m_a(t) = m_s(t) = 10$ units and with 80 units in the waiting state. The values are shown in Figure 8. Since the solution remains in Region 2, a relatively large value of DT can be used. In this example DT = 1. The model acts like a classical reliability type model in that the failure rate is strong enough, relative to the capability for service, that the lack of available units falls until a relatively low steady state level is reached.

Example 4. This example illustrates an extremely rapid transition from a Region 1 to a Region 2 type of solution. The parameter values are: $\mu = .4$, $\lambda = .4$, and $r = 10$. The initial values are: $a_1 = 89$, $s_1 = 5$, and $w_1 = 6$ for a total of 100 units. Because of the rapid transition between regions a small value of DT is required so as to not miss or greatly distort the transition affect. A value of DT = .001 time

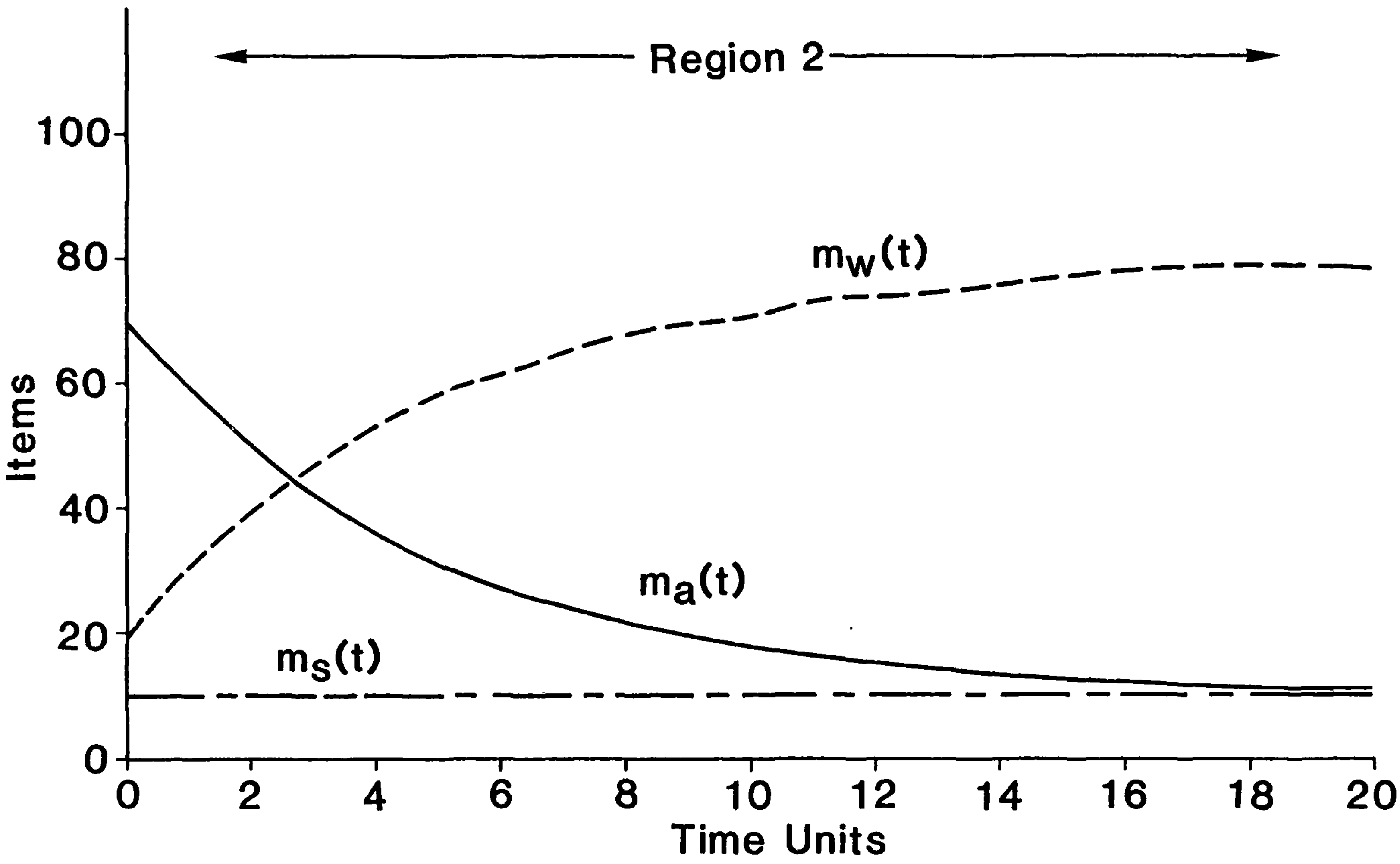

Figure 8

Example 3

units was used for this example. The values shown in Figure 9 give the Region 1, Case 1 ($D = -.6<0$) solution which shifts to a Region 2 solution at $T = .035$ time units. The values of the descriptive functions $m_a(t)$, $m_s(t)$, and $m_w(t)$ do not change much over this short time interval. Though the example is given to illustrate how fast the model can shift from one solution form to another, it should be noted that in applications, the time units may be such as to give practical interest to relatively short time periods. This is discussed more fully in Chapter 5. The most important aspect of this example is the need for using a very small DT value. Otherwise the true nature of the solution is lost and a numerical solution is obtained that does not properly represent the nature of the model solution for this set of parameters and initial conditions.

Example 5. A less rapid transition from a Region 1 to a Region 2 solution is illustrated by this example. The parameter values are $\mu = .4$, $\lambda = .2$, and $r = 10$. Initial conditions are $a_1 = 100$, $s_1 = 0$, and $w_1 = 0$ so that at the start of the analysis all units are available. Though the transition between regions is not as rapid as for Example 4 it is sufficiently fast that care is required in the selection of the evaluation time interval if correct Region 2 solutions are to be obtained. The value used is $DT = .05$ time units. Transition from Region 1 to Region 2 takes place at $T = .5$ time units and by 2 time units, the service facility has almost reached its capacity of 10 units. Though the example does not go completely to a steady state in 2 time units, the model descriptors have come close to steady state values by this time. Figure 10 shows the results and indicates extrapolated steady state values of $m_a(t) = 68$, $m_s = 10$, and $m_w = 22$.

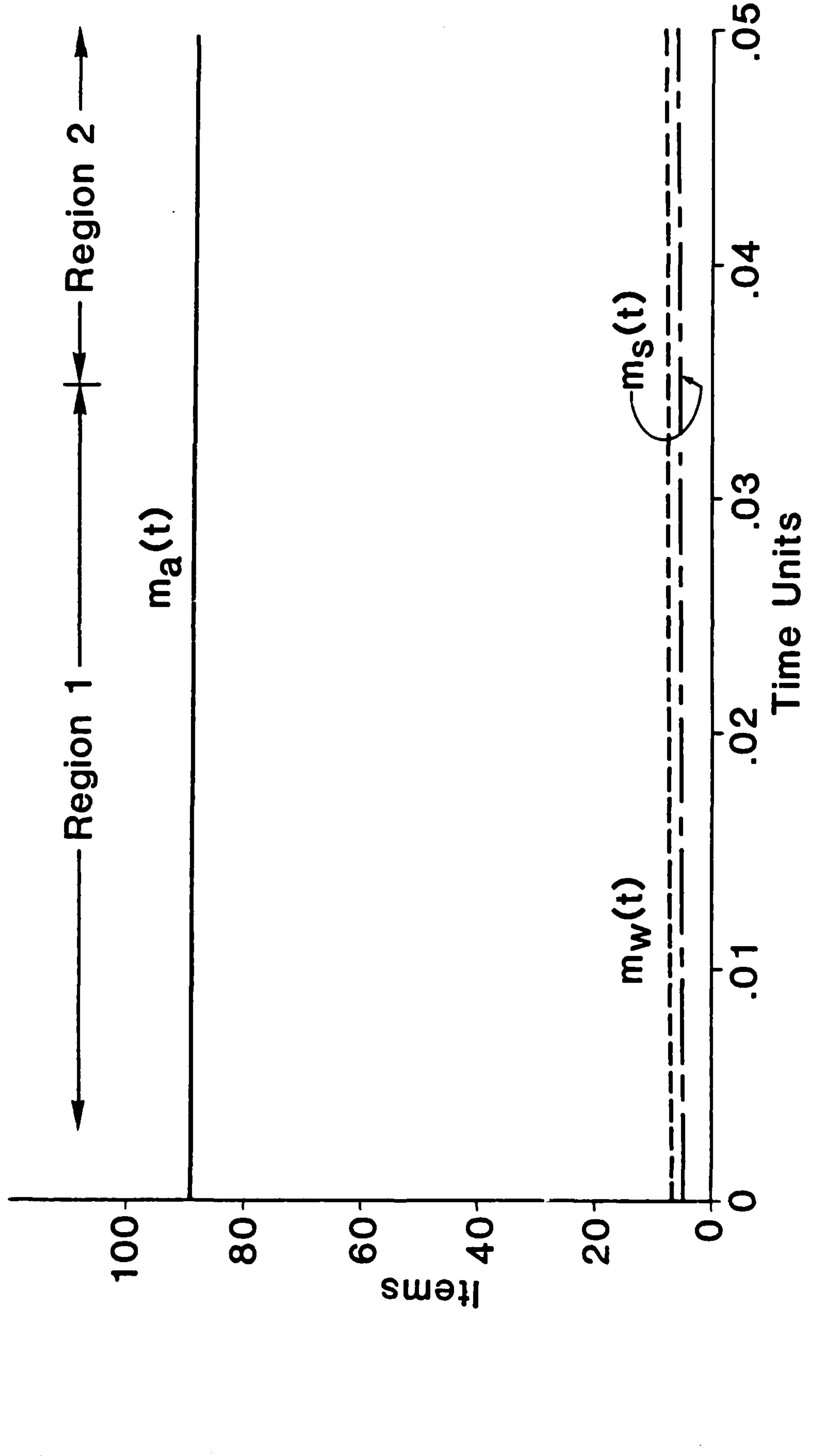

Figure 9

Example 4

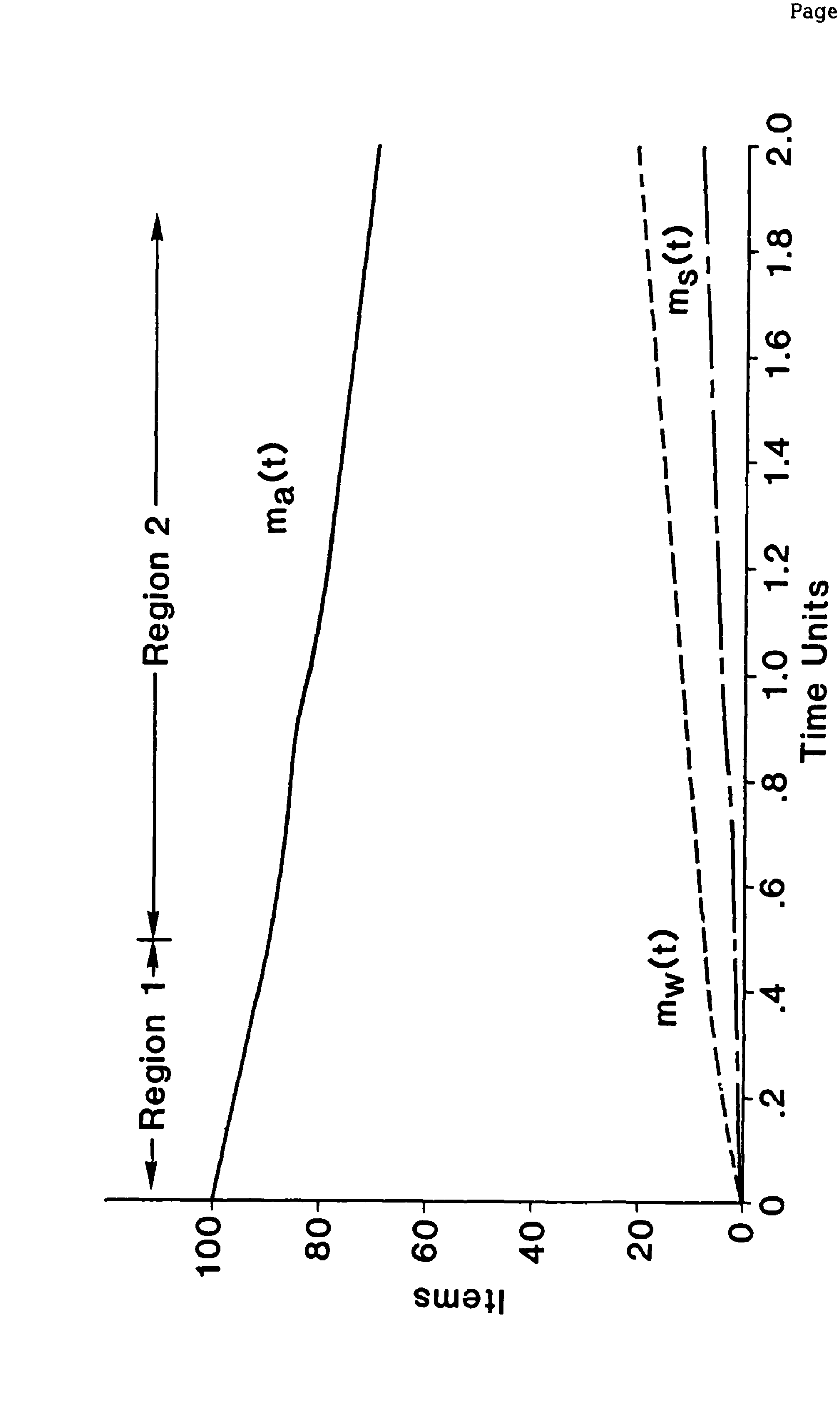

Figure 10

Example 5

CHAPTER 5. BULK AVAILABILITY WARRANTIES

Over the past several years, customers, such as the U.S. Government, have become increasingly concerned with the satisfactory operation of equipment after it has been delivered. Acceptance test procedures insure that equipment meets contractual requirements at the time of delivery but provide little indication of successful subsequent operation. Reliability tests that form part of many such procedures have minimal predictive value due to such factors as lack of statistically significant data, intrinsic variability in operational conditions, and limited periods for observation of equipment before completion of acceptance procedures. This situation has led to increased interest in warranty policies that may be included as part of contract specification. In particular, the concept of the Reliability Improvement Warranty (RIW) provides a means for the contractor to assume some of the risk associated with the customer's acceptance of equipment which subsequently fails to operate properly. The development of RIW models, their analysis, and their application to contract definition has brought about a well-defined field of study blending classical warranty concepts with stochastic and other mathematical techniques from the field of Reliability (7).

Such warranties may be considered from several points of view. They may be considered as motivation for a contractor to strive for high reliability products. They can be a means for clearly specifying the division of risk assumption between contractor and customer due to product failure. They can aslo provide a range of design cost trade-offs in which a contractor can choose between spending money on producing a more reliable product or on paying warranty costs (4).

An immediate extension of the RIW concept is to consider warranties from the point of view of availability rather than reliability. This places the focus on the availability of a product for use when needed which can be achieved by improving its reliability and/or providing improved repair/replace options. Availability warranties seem particularly attractive to the customer. However, they may be viewed as undesirable contract obligations by a contractor. By combining the concepts of incentive contracts and availability warranties a wide range of contract structures can be formulated in which a combination of risk assumption and profitability can be achieved (8).

Much of the study of availability warranties has considered the availability of a single item or system which is subject to failure and repair/replace service over the duration of the warranty period (9), (3). This represents a classical viewpoint in the study of availability. The present paper deals with another kind of availability consideration which considers a large number of units rather than a single item. Each of the single units is relatively minor in its required function and availability is specified by having a minimum number of units available.

In this study, the availability concepts are those of bulk availability rather than individual item availability. Therefore a deterministic model for bulk availability, as defined in Chapter 1, has been used as a basis for anlaysis rather than the more common approach based on stochastic models. This procedure seems analogous to the role of thermodynamics in describing gross characteristics of matter that require statistical mechanics for their detailed investigation. This analogy provided the guidelines for development of the present bulk availability model.

Bulk availability deals with such things as a large work force (manpower availability), fleets of equipment, or graceful degradation of complex systems. An example is a military electronic communications device which may be supplied in several thousand units. A particular group may have 100 such devices assigned to it and require 80 for completely satisfactory operation. This would allow 20 of the devices to be either undergoing service or waiting for service. If more than 20 were out of order, the bulk system would be, to some degree, unavailable. If an availability warranty is in effect in such a situation, the contractor would have to pay penalty costs specified by the warranty contract. Such costs can be viewed in various ways: as out-and-out payment of penalty dollars, as a requirement to spend money on improved repair/replace facilities, or as the free (or reduced cost) supply of one or more additional units by the contractor.

This report has described the deterministic bulk availability model and how analyses can be carried out using the model. It gave some examples of the behavior of the bulk availability models. How they specify availability as a particular level of available units will be described in this chapter.

Bulk availability warranties are defined as costs incurred by the contractor when the number of available units falls below contractually specified levels. The cost trade-offs available to a contractor are illustrated by the examples. Conclusions about the utilization and applicability of the bulk availability warranty concept are discussed in the final chapter of the report.

The most immediate application of the model to a bulk availability warranty analysis is when a penalty fee is charged to the contractor

whenever the number of available units falls below a specified level. This kind of response to a lowered level of availability has no effect on the time functions of the model. When necessary, the fee is paid and the model continues to represent the bulk availability situation. Payments of this kind can be as a single penalty or can be tied to the length of time over which the availability is too low. In situations such as those shown in Figures 8 and 10, the availability function decreases monotonically. In such a case if the function falls below the specified availability level before the warranty time has expired, a penalty will be incurred by the contractor. If it is of the time duration type it is clearly desirable for the low level of availability to be reached as near to the end of the warranty period as possible. This shows the value of a time dependent penalty to the customer. It can also be used tc help the contractor in both product design and contract structure decisions. In a situation like that shown in Figure 7 the availability is increasing. In such a case, the contractor might incur a warranty penalty initially but would subsequently be free of warranty payments. This indicates the importance of design considerations. A high enough initial state for availability would prevent any warranty costs. However, the cost to achieve such an initial level could exceed the initial warranty cost and therefore be undesirable. Another consideration is the increasing availability function. This kind of product performance may be better than required, at a cost to the contractor. An alternative design, with lower cost and lower availability might serve both parties better, giving satisfactory performance to the customer and higher profit to the contractor. Figure 6 shows a variation in the availability function. In a case such as this, the proper level for availability may be in effect at

some periods of time and not during other periods. This is an interesting case from the warranty point of view. It allows a wide range of contract stipulations that might not be considered without a knowledge of expected behavior of this kind. Thus, in this type of bulk availability model, it is particularly important to have some kind of qualitative guidelines for the behavior of the availability function, as provided by the model analysis.

A more realistic type of warranty obligation is to provide some form of correction for the occurrence of low value of the availability function. When the availability function falls below a specified value the contractor may be required to bring the number of available units up to a level that will yield a satisfactory value. This will require replacement and/or repair of units. An action of this kind interrupts the time functions of the model analysis. When it is desired to employ this kind of response to a warranty obligation, the mathematical model must be interrupted. A new set of initial conditions are defined as a combination of what was being specified by the model upon interruption and actions taken by the contractor before resumption of the model analysis. Some way to measure time must also be selected so as to represent time in a useful way. Most likely the time should be continued to be measured from the beginning of the warranty period. In this case, the time between model interruption and the new start should be included in the total time description. This time measures the time required to bring the availability up to the desired level. It is seen that the repair/replace procedure for satisfying warranty obligations can be studied within the framework of the bulk availability model of this paper by making the kinds of modifications to the analysis described above.

CHAPTER 6. CONCLUSIONS AND EXTENSIONS

A deterministic model for bulk availability has been described in this report. It has a fixed total population and what may be called an r-server type repair queue system. In some ways the model appears similar to a stochastic queuing model with these characteristics. However, the bulk model has gross state transition values rather than individual transition effects. This makes it more like a closed system dam type process than a particle queue process. Though the availability level curve as a function of time looks much like a mean availability curve for various queue processes, it does not seem to follow in any direct way from such processes. This is most likely due to the bulk transition effect which prevents the application of the basic non-cascade assumption necessary for the formulation of most particle queue models. Research in a dam type process expressing properties similar to the present model might be of theoretical interest but is considered outside the scope of the research described here.

The model description and analysis described in this report shows that:

- The bulk availability model can be employed as a useful tool in trade-off analyses between manufacturing, logistic, and warranty costs. By using several versions of a proposed model, with different parameter values, a set of possible availability functions can be generated. This kind of study can give guidelines for determining the best values or ranges of values to aim for in designing a product to meet selected cost trade-off values.

- The deterministic model is simpler to formulate, interpret, and explain than corresponding stochastic models. This is particularly true

in the present case where the appropriate stochastic model seems to be of the finite dam type in which the mathematics is more involved than for queue type models. Ease in application and in intuitive relevence between parameter values and results, such as availability levels, is particularly useful in contract negotiations where one party may be required to explain or justify its analyses as part of its justification for cost trade-off decisions.

· The closed form of direct solution for this kind of model allows the consideration of numerical results over any desirable time period, subject only to considerations of transition between solution regions. The examples given in Chapter 4 illustrate the ease with which numerical solutions can be obtained using readily available computing resources.

Though the calculations required are sufficiently involved that the model could not be used without computer implementation, the computer time and storage requirements are small. For example, 20 steps of the model run in what appear to be no time to an interactive terminal user on a standard computer installation.

· The bulk availability model can be given broader interpretations. For example, if it is to be used as a model for manpower availability the quantity λ does not represent manufacturing effort, it relates to the profile of people recruited into the manpower pool. Thus λ can be related to recruitment effort costs such as extent of physical and mental examination of candidates. It can also be related to the level of physical or mental condition that will be used in accepting or rejecting candidates.

The bulk availability model given in this report could be made more widely applicable if it was extended to allow time variation in some

or all of the model parameters λ, μ, and r. In the present model, the most usual form of solution is to move to Region 2 and remain there tending toward, and usually reaching rather soon, a steady state condition with the service facility full. If the parameters change with time a more dynamic model would result which would allow transitions back to Region 1. Such a dynamic model is felt to be a reasonable one in modeling situations in which a contractor can change parameter values or operating conditions cause them to change during the warranty period.

CHAPTER 7. REFERENCES

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(3) C. Marshall, Incentivising Availability Warranties, Report POLY-EE 80-002, Polytechnic Institute of New York, New York, 1980.

(4) C. Marshall, "Design Trade-Offs in Availability Warranties," Proc. Annual Reliability and Maintainability Symposium (Philadelphia), Jan. 1981.

(5) W.R. Blischke, E.M. Scheuer, "Applications of Renewal Theory in Analysis of the Free-Replacement Warranty," Naval Research Logistics Quarterly, 28 (1981), pp. 193-204.

(6) C. Marshall, "Warranty Definition for Bulk Availability," Proc. Annual Reliability and Maintainability Symposium (Orlando), Jan. 1983.

(7) F.M. Biedenweg, Warranty Policies: Consumer Value vs. Manufacturer Costs, Dept. Operations Research, Stamford University, Technical Report 198, April 1981.

(8) C. Marshall, "Incentivising Reliability Improvement Warranties," Proc. Ninth Annual DOD/FAI Acquisition Research Symposium (Annapolis), June 1980.

(9) W.R. Blischke, E.M. Scheuer, "Calculation of the Cost of Warranty Policies as a Function of Estimated Life Distributions," Naval Research Logistics Quarterly, 22 (1975), pp. 681-696.

APPENDIX

Computer Program

A listing of the FORTRAN computer program used to carry out the numerical evaluation of the model is given below. It is one particular realization of the flow diagram shown in Figure 5. The main reason for giving this listing is to illustrate the simple programming required to implement the model in terms of numerical calculations.

```
      WRITE(6,*) 'INPUT DT AND TF'
      READ(5,*) DT,TF
      WRITE(9,300) DT,TF
      WRITE(6,*) 'INPUT A1,W1, AND S1'
      READ(5,*) A1,W1,S1
      WRITE(9,310) A1,W1,S1
      WRITE(6,*) 'INPUT XMU,XLM,AND R'
      READ(5,*) XMU,XLM,R
      WRITE(9,320) XMU,XLM,R
      WRITE(6,*) 'INPUT K'
      READ(5,*) K
      WRITE(9,360) K
      T=0.
      TOTT=0.
      TLOC=0.
      KTR=0
      AA=R-S1+XMU*S1
      IF(W1.GT.AA) GO TO 20
      KR=1
   60 CONTINUE
      D=(1.-XLM)*(1.-XLM)-2.*XMU*(1.+XLM)+XMU*XMU
      ALP=(1.+XLM+XMU)/2.
      AB=ABS(D/4.)
      BTA=SQRT(AB)
C       WRITE(9,350) D,ALP,AB,BTA
      IF(D.EQ.0.) GO TO 70
      IF(D.GT.0.) GO TO 80
      COM=(ALP*ALP)+(BTA*BTA)
      A=(S1+W1+A1)/COM
      C1=(XMU*S1*(1.+XLM)-XLM*(A1+W1))*EXP(-ALP*T)
      C2=XMU*S1*(1.+XLM*XLM-XMU-XMU*XLM)+W1*(2.*XMU-XLM*XLM
     1+XLM+XLM*XMU)
      C3=-XLM*A1*(1.+XLM+XMU)
      XMS=XLM*A+(C1*COS(BTA*T)/COM)+((C2+C3)*EXP(-ALP*T)*SIN(BTA*T)
     1/(2.*BTA*COM))
      C4=(-(S1+W1)*XMU+A1*XLM*(1.+XMU))*EXP(-ALP*T)
      C5=S1*XMU*(XLM+XMU+2.*XLM*XMU-1.)-W1*XMU*(1.+XLM+XMU)
      C6=A1*XLM*(1.+XMU*XMU-XLM-XLM*XMU)
      XMA=XMU*A+(C4*COS(BTA*T)/COM)+((C5+C6)*EXP(-ALP*T)*SIN(BTA*T)
     1/(2.*BTA*COM))
      C7=(W1*(XLM+XMU)-XLM*XMU*(S1+A1))*EXP(-ALP*T)
      C8=W1*(XLM*XLM+XMU*XMU-XLM-XMU)+A1*XLM*(2.*XLM+XMU+XLM*XMU
     1-XMU*XMU)
      C9=-S1*XLM*XMU*(1.+XMU+XLM)
      XMW=XMU*XLM*A+(C7*COS(BTA*T)/COM)+((C8+C9)*EXP(-ALP*T)*SIN(BTA*T)
     1/(2.*BTA*COM))
      GO TO 100
```

```
   7Ø B=(S1+W1+A1)/(ALP*ALP)
      C=W1+S1*(1.+XLM-ALP)-XLM*ALP*B
      C1=C*T*EXP(-ALP*T)
      C2=(S1-XLM*B)*EXP(-ALP*T)
      XMS=XLM*B+C1+C2
      C3=(-S1-W1+XLM*(2.*ALP-XLM)*B)
      XMA=XMU*B+((XMU*C3*EXP(-ALP*T))/((XLM-ALP)*(XLM-ALP)))
     1-(XMU*C1/(ALP-XMU))
      C4=(W1-XLM*XMU*B)*EXP(-ALP*T)
      XMW=XMU*XLM*B+C4+(XMU-ALP)*C1
      GO TO 1ØØ
   8Ø TH1=BTA-ALP
      TH2=-BTA-ALP
      H1=(A1+W1+S1)*XLM/(XLM+XMU+XLM*XMU)
      C1=S1*(1.+XLM)+W1
      C2=(A1+W1+S1)*XLM/(2.*BTA)
      H2=((S1*TH1+C1)/(2.*BTA))+(C2/TH1)
      H3=((-S1*TH2-C1)/(2.*BTA))-(C2/TH2)
      XMS=H1+H2*EXP(TH1*T)+H3*EXP(TH2*T)
      XMW=XMU*H1+H2*(TH1+XMU)*EXP(TH1*T)+H3*(TH2+XMU)*EXP(TH2*T)
      XMA=(H1*XMU/XLM)+((H2*XMU*EXP(TH1*T))/(XLM+TH1))
     1+((H3*XMU*EXP(TH2*T))/(XLM+TH2))
      GO TO 1ØØ
   2Ø KR=2
   5Ø CONTINUE
      XMS=(S1-R)*EXP(-T)+R
      C1=XMU*(S1-R)/(XLM-1.)
      XMA=(XMU*R*(1.-EXP(-XLM*T))/XLM)+C1*(EXP(-T)-EXP(-XLM*T))
     1+A1*EXP(-XLM*T)
      C2=A1+W1+S1-R-(XMU*R/XLM)
      C3=(XMU*(R-XLM*S1)/(XLM-1.)+(XLM*A1))/XLM
      C4=(S1-R)*(XLM+XMU-1.)/(XLM-1.)
      XMW=C2-C3*EXP(-XLM*T)-C4*EXP(-T)
      GO TO 1ØØ
  1ØØ TLOC=TOTT+T
      KTR=KTR+1
      IF(KTR.LE.K) GO TO 11Ø
      WRITE(9,33Ø) TLOC,XMA,XMS,XMW
      WRITE(9,34Ø) KR,T,TOTT
      KTR=Ø
  11Ø CONTINUE
      IF(TLOC.GE.TF) GO TO 999
      IF(XMW.LT.R-XMS+XMU*XMS) GO TO 2ØØ
      IF(KR.EQ.2) GO TO 15Ø
      A1=XMA
      S1=XMS
      W1=XMW
      TOTT=T+TOTT
      T=DT
      KR=2
      GO TO 5Ø
  15Ø T=T+DT
      GO TO 5Ø
  2ØØ IF(KR.EQ.1) GO TO 25Ø
      A1=XMA
      S1=XMS
      W1=XMW
      TOTT=T+TOTT
      T=DT
      KR=1
      GO TO 6Ø
  25Ø T=T+DT
      GO TO 6Ø
  999 STOP
  3ØØ FORMAT(5X,'DT=',F1Ø.5,5X,'TF=',F1Ø.5)
  31Ø FORMAT(5X,'A1=',F1Ø.5,5X,'W1=',F1Ø.5,5X,'S1=',F1Ø.5)
  32Ø FORMAT(5X,'XMU=',F1Ø.5,5X,'XLM=',F1Ø.5,5X,'R=',F1Ø.5)
  33Ø FORMAT(5X,'TLOC=',F1Ø.5,5X,'XMA=',F1Ø.5,5X,'XMS=',F1Ø.5,
     15X,'XMW=',F1Ø.5)
  34Ø FORMAT(5X,'KR=',I3,5X,'T=',F1Ø.5,5X,'TOTT=',F1Ø.5)
C 35Ø FORMAT(5X,'TEST    ',4F1Ø.5)
  36Ø FORMAT(5X,'K=   ',I5)
      END
```

Numerical Data for Examples

The numerical data, produced by the computer program listed above, are given below for the examples given in Chapter 4 of the report. They are listed by Example.

Example shown in Figure 7:

```
DT=    1.00000       TF=  20.00000
A1=  89.00000        W1=    6.00000       S1=    5.00000
XMU=    0.40000       XLM=    0.01000       R=  10.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    0.00000        XMA=  89.00001        XMS=    5.00000      XMW=    6.00000
KR=  1       T=    0.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    1.00000        XMA=  90.56210        XMS=    6.65261      XMW=    2.77530
KR=  1       T=    1.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    2.00000        XMA=  92.24326        XMS=    6.15701      XMW=    1.59975
KR=  1       T=    2.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    3.00000        XMA=  93.59267        XMS=    5.20056      XMW=    1.17677
KR=  1       T=    3.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    4.00000        XMA=  94.57150        XMS=    4.40021      XMW=    1.02830
KR=  1       T=    4.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    5.00000        XMA=  95.24976        XMS=    3.77152      XMW=    0.97873
KR=  1       T=    5.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    6.00000        XMA=  95.70903        XMS=    3.32699      XMW=    0.96393
KR=  1       T=    6.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    7.00000        XMA=  96.01637        XMS=    3.02283      XMW=    0.96080
KR=  1       T=    7.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    8.00000        XMA=  96.22060        XMS=    2.81822      XMW=    0.96119
KR=  1       T=    8.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=    9.00000        XMA=  96.35582        XMS=    2.68183      XMW=    0.96236
KR=  1       T=    9.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  10.00000        XMA=  96.44513        XMS=    2.59136      XMW=    0.96346
KR=  1       T=  10.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  11.00000        XMA=  96.50417        XMS=    2.53152      XMW=    0.96432
KR=  1       T=  11.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  12.00000        XMA=  96.54308        XMS=    2.49230      XMW=    0.96493
KR=  1       T=  12.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  13.00000        XMA=  96.56873        XMS=    2.46593      XMW=    0.96535
KR=  1       T=  13.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  14.00000        XMA=  96.58565        XMS=    2.44873      XMW=    0.96563
KR=  1       T=  14.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
TLOC=  15.00000        XMA=  96.59679        XMS=    2.43739      XMW=    0.96582
KR=  1       T=  15.00000        TOTT=    0.00000
TEST        0.33210    0.70500    0.08303    0.28814
```

```
TLOC=  16.00000      XMA=  96.69415      XMS=    2.42991      XMW=   0.96594
KR=  1      T=  16.00000      TOTT=   0.00000
TEST      0.33210   0.70500   0.08303   0.28814
TLOC=  17.00000      XMA=  96.65399      XMS=    2.42499      XMW=   0.96602
KR=  1      T=  17.00000      TOTT=   0.00000
TEST      0.33210   0.70500   0.08303   0.28814
TLOC=  18.00000      XMA=  96.61219      XMS=    2.42174      XMW=   0.96608
KR=  1      T=  18.00000      TOTT=   0.00000
TEST      0.33210   0.70500   0.08303   0.28814
TLOC=  19.00000      XMA=  96.61430      XMS=    2.41960      XMW=   0.96611
KR=  1      T=  19.00000      TOTT=   0.00000
```

Example shown in Figure 8:

```
DT=    1.00000     TF=  20.00000
A1=  70.00000      W1=  20.00000        S1=  10.00000
XMU=   0.20000     XLM=   0.20000        R=  10.00000
TLOC=   0.00000      XMA=  70.00000      XMS=  10.00000      XMW=  20.00000
KR=  2      T=   0.00000      TOTT=   0.00000
TLOC=   1.00000      XMA=  59.12384      XMS=  10.00000      XMW=  30.87615
KR=  2      T=   1.00000      TOTT=   0.00000
TLOC=   2.00000      XMA=  50.21920      XMS=  10.00000      XMW=  39.78080
KR=  2      T=   2.00000      TOTT=   0.00000
TLOC=   3.00000      XMA=  42.92870      XMS=  10.00000      XMW=  47.07130
KR=  2      T=   3.00000      TOTT=   0.00000
TLOC=   4.00000      XMA=  36.95974      XMS=  10.00000      XMW=  53.04026
KR=  2      T=   4.00000      TOTT=   0.00000
TLOC=   5.00000      XMA=  32.07277      XMS=  10.00000      XMW=  57.92723
KR=  2      T=   5.00000      TOTT=   0.00000
TLOC=   6.00000      XMA=  28.07165      XMS=  10.00000      XMW=  61.92835
KR=  2      T=   6.00000      TOTT=   0.00000
TLOC=   7.00000      XMA=  24.79582      XMS=  10.00000      XMW=  65.20419
KR=  2      T=   7.00000      TOTT=   0.00000
TLOC=   8.00000      XMA=  22.11379      XMS=  10.00000      XMW=  67.88621
KR=  2      T=   8.00000      TOTT=   0.00000
TLOC=   9.00000      XMA=  19.91793      XMS=  10.00000      XMW=  70.08207
KR=  2      T=   9.00000      TOTT=   0.00000
TLOC=  10.00000      XMA=  18.12012      XMS=  10.00000      XMW=  71.87988
KR=  2      T=  10.00000      TOTT=   0.00000
TLOC=  11.00000      XMA=  16.64819      XMS=  10.00000      XMW=  73.35181
KR=  2      T=  11.00000      TOTT=   0.00000
TLOC=  12.00000      XMA=  15.44308      XMS=  10.00000      XMW=  74.55692
KR=  2      T=  12.00000      TOTT=   0.00000
TLOC=  13.00000      XMA=  14.45641      XMS=  10.00000      XMW=  75.54359
KR=  2      T=  13.00000      TOTT=   0.00000
TLOC=  14.00000      XMA=  13.64860      XMS=  10.00000      XMW=  76.35139
KR=  2      T=  14.00000      TOTT=   0.00000
TLOC=  15.00000      XMA=  12.98722      XMS=  10.00000      XMW=  77.01278
KR=  2      T=  15.00000      TOTT=   0.00000
TLOC=  16.00000      XMA=  12.44573      XMS=  10.00000      XMW=  77.55427
KR=  2      T=  16.00000      TOTT=   0.00000
TLOC=  17.00000      XMA=  12.00240      XMS=  10.00000      XMW=  77.99760
KR=  2      T=  17.00000      TOTT=   0.00000
TLOC=  18.00000      XMA=  11.63942      XMS=  10.00000      XMW=  78.36058
KR=  2      T=  18.00000      TOTT=   0.00000
TLOC=  19.00000      XMA=  11.34225      XMS=  10.00000      XMW=  78.65775
KR=  2      T=  19.00000      TOTT=   0.00000
TLOC=  20.00000      XMA=  11.09894      XMS=  10.00000      XMW=  78.90106
KR=  2      T=  20.00000      TOTT=   0.00000
```

Example shown in Figure 9:

```
DT=    0.00100     TF=   0.05000
A1=  89.00000      W1=   6.00000        S1=   5.00000
XMU=   0.40000     XLM=   0.40000        R=  10.00000
K=        2
TLOC=   0.00200      XMA=  88.93224      XMS=   5.00805      XMW=   6.05911
KR=  1      T=   0.00200      TOTT=   0.00000
TLOC=   0.00500      XMA=  88.83220      XMS=   5.02034      XMW=   6.14746
KR=  1      T=   0.00500      TOTT=   0.00000
TLOC=   0.00800      XMA=  88.73163      XMS=   5.03239      XMW=   6.23543
KR=  1      T=   0.00800      TOTT=   0.00000
```

```
TLOC=   0.01100     XMA=  88.63132     XMS=   5.04568     XMW=   6.32301
KR=  1      T=   0.01100    TOTT=   0.00000
TLOC=   0.01400     XMA=  88.53100     XMS=   5.05072     XMW=   6.41020
KR=  1      T=   0.01400    TOTT=   0.00000
TLOC=   0.01700     XMA=  88.43098     XMS=   5.07200     XMW=   6.49702
KR=  1      T=   0.01700    TOTT=   0.00000
TLOC=   0.02000     XMA=  88.33102     XMS=   5.08552     XMW=   6.58346
KR=  1      T=   0.02000    TOTT=   0.00000
TLOC=   0.02300     XMA=  88.23119     XMS=   5.09929     XMW=   6.66952
KR=  1      T=   0.02300    TOTT=   0.00000
TLOC=   0.02600     XMA=  88.13151     XMS=   5.11330     XMW=   6.75520
KR=  1      T=   0.02600    TOTT=   0.00000
TLOC=   0.02900     XMA=  88.03196     XMS=   5.12755     XMW=   6.84050
KR=  1      T=   0.02900    TOTT=   0.00000
TLOC=   0.03200     XMA=  87.93254     XMS=   5.14204     XMW=   6.92543
KR=  1      T=   0.03200    TOTT=   0.00000
TLOC=   0.03500     XMA=  87.83325     XMS=   5.15659     XMW=   7.01016
KR=  2      T=   0.00300    TOTT=   0.03200
TLOC=   0.03800     XMA=  87.73412     XMS=   5.17110     XMW=   7.09479
KR=  2      T=   0.00600    TOTT=   0.03200
TLOC=   0.04100     XMA=  87.63511     XMS=   5.18556     XMW=   7.17933
KR=  2      T=   0.00900    TOTT=   0.03200
TLOC=   0.04400     XMA=  87.53624     XMS=   5.19999     XMW=   7.26378
KR=  2      T=   0.01200    TOTT=   0.03200
TLOC=   0.04700     XMA=  87.43750     XMS=   5.71435     XMW=   7.34814
KR=  2      T=   0.01500    TOTT=   0.03200
TLOC=   0.05000     XMA=  87.33891     XMS=   5.22870     XMW=   7.43241
KR=  2      T=   0.01800    TOTT=   0.03200
```

Example shown in Figure 10.

```
DT=    0.05000      TF=    2.00000
A1= 100.00000       W1=    0.00000      S1=    0.00000
XMU=   0.40000      XLM=   0.20000      R=   10.00000
K=        2
TLOC=   0.10000     XMA=  98.02113     XMS=   0.09482     XMW=   1.88404
KR=  1      T=   0.10000    TOTT=   0.00000
TLOC=   0.25000     XMA=  95.14156     XMS=   0.54749     XMW=   4.31094
KR=  1      T=   0.25000    TOTT=   0.00000
TLOC=   0.40000     XMA=  92.38303     XMS=   1.29569     XMW=   6.32127
KR=  1      T=   0.40000    TOTT=   0.00000
TLOC=   0.55000     XMA=  89.75717     XMS=   2.26621     XMW=   7.97661
KR=  1      T=   0.55000    TOTT=   0.00000
TLOC=   0.70000     XMA=  87.27042     XMS=   3.32983     XMW=   9.39975
KR=  2      T=   0.10000    TOTT=   0.60000
TLOC=   0.85000     XMA=  84.91630     XMS=   4.25893     XMW=  10.82477
KR=  2      T=   0.25000    TOTT=   0.60000
TLOC=   1.00000     XMA=  82.68272     XMS=   5.05861     XMW=  12.25866
KR=  2      T=   0.40000    TOTT=   0.60000
TLOC=   1.15000     XMA=  80.55904     XMS=   5.74691     XMW=  13.69404
KR=  2      T=   0.55000    TOTT=   0.60000
TLOC=   1.30000     XMA=  78.53539     XMS=   6.33933     XMW=  15.12478
KR=  2      T=   0.70000    TOTT=   0.60000
TLOC=   1.45000     XMA=  76.60503     XMS=   6.84923     XMW=  16.54573
KR=  2      T=   0.85000    TOTT=   0.60000
TLOC=   1.60000     XMA=  74.75923     XMS=   7.28811     XMW=  17.95266
KR=  2      T=   1.00000    TOTT=   0.60000
TLOC=   1.75000     XMA=  72.99205     XMS=   7.66585     XMW=  19.34209
KR=  2      T=   1.15000    TOTT=   0.60000
TLOC=   1.90000     XMA=  71.29783     XMS=   7.99098     XMW=  20.71119
KR=  2      T=   1.30000    TOTT=   0.60000
TLOC=   2.05000     XMA=  69.67152     XMS=   8.27082     XMW=  22.05766
KR=  2      T=   1.45000    TOTT=   0.60000
```

END

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